DERIVED MODELS OF MICE BELOW THE LEAST FIXPOINT OF THE SOLOVAY SEQUENCE

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Abstract. We introduce a mouse whose derived model satisfies $\mathrm{AD}_{\mathbb{R}} + \Theta \geq \theta_{\aleph_2}$. More generally, we will introduce a class of large cardinal properties yielding mice whose derived models can satisfy properties as strong as $\mathrm{AD}_{\mathbb{R}} + \Theta = \theta_\Theta$.

In descriptive inner model theory consistency strength is graded in terms of the Solovay sequence; the majority of set theorists on the other hand work within the large cardinal hierarchy. This necessitates a translation procedure.

In [7] Steel introduced a method of how to canonically associate to an (ω_1+1)-iterable premouse M and $\lambda \in M$ a limit of M-Woodin cardinals a model of determinacy $D(M,\lambda)$, the derived model of M at λ, provided the pair (M,λ) is suitably well-behaved.

In his PhD thesis ([1]) Closson introduced a large cardinal notion called the hyperstrong cardinal and showed that the sharp M for a limit of hyperstrong cardinals and Woodin cardinals λ, if it exists, has a canonical derived model at λ and $D(M,\lambda)$ satisfies that $\Theta > \theta_{\omega_1}$.

Steel in [7] asked if there is an analogous mouse for "$\Theta > \theta_{\omega_2}$" (Remark 8.2 (c)). Here we will use a slightly stronger version of the hyperstrong cardinal to define a putative $M^{\#}_{\theta_{\omega_2}}$, by which we mean the least mouse M with some λ such that a canonical derived model of M at λ exists and $D(M,\lambda)$ satisfies $\Theta = \theta_{\omega_2}$. We will show:

THEOREM 1. *Assume that $M^{\#}_{\theta_{\omega_2}}$ exists (see Definition 28) and has a good (ω_1+1)-iteration strategy. Let λ be $M^{\#}_{\theta_{\omega_2}}$'s largest limit of Woodin cardinals. Then $D(M^{\#}_{\theta_{\omega_2}},\lambda)$ exists and satisfies $\Theta \geq \theta_{\omega_2}$.*

We will also present a variant of this construction whose derived model will satisfy $\Theta > \theta_{\omega_2}$ as in Steel's original question.

We will not prove that $M^{\#}_{\theta_{\omega_2}}$ is actually the least mouse satisfying the conclusion of Theorem 1 as that requires a different tool set (see [13]) and thus lies outside the scope of this paper.

Closson and Steel also proved that $M^{\#}_{wlim}$, the sharp for a Woodin cardinal that is itself a limit of Woodin cardinals, has a canonical derived model at its largest Woodin cardinal λ and $D(M^{\#}_{wlim},\lambda)$ satisfies $\Theta = \theta_\Theta$.

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We will introduce a more general large cardinal property related to the hyperstrong cardinal where the degree of strongness is no longer tied to an ordinal but to a pre-wellorder on V_λ. Using this we can produce a mouse we call $M^{\#}_{\theta_\Theta}$. We show:

THEOREM 2. *Assume that $M^{\#}_{\theta_\Theta}$ exists (see Definition 42) and has a good (ω_1+1)-iteration strategy. Let λ be $M^{\#}_{\theta_\Theta}$'s largest limit of Woodin cardinals. Then $D(M^{\#}_{\theta_\Theta}, \lambda)$ exists and satisfies $\Theta = \theta_\Theta$.*

Note that apart from the framework provided by [7] our methods will be quite different from those used by Closson and Steel. Our methods are mostly rooted in the works of Sargsyan on HOD-mice. ([4]).

The paper will be organized as follows: section 1 will review results from [7] needed for our argument; section 2 will do the same for [4]; section 3 will introduce the notion of hyperstrongness used to define $M^{\#}_{\theta_{\omega_2}}$; section 4 will introduce that mouse and show that it is suitably well-behaved in the sense of [7]; section 5 will be dedicated to the proof of Theorem 1; section 6 will introduce nonlinear degrees of hyperstrongness, define $M^{\#}_{\theta_\Theta}$ and prove Theorem 2; section n for $6 < n < \omega$ does not exist.

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§1. Introduction. Let M be a transitive model of $\mathrm{ZF} + \mathrm{AD}^+$. Θ^M refers to the supremum of pre-wellorders on $\mathbb{R}$ in M. We will refine this notation. Define by induction a sequence of ordinals $\langle \theta_\alpha : \alpha \leq \beta \rangle$:

- $\theta_0 := \sup\{\gamma | \exists f \in \mathrm{OD}^M : f : \mathbb{R} \twoheadrightarrow \gamma\}$;
- $\theta_{\alpha+1} := \sup\{\gamma | \exists f \in \mathrm{OD}^M_A : f : \mathbb{R} \twoheadrightarrow \gamma\}$ if there is $A \in (\mathcal{P}(\mathbb{R}))^M$ with Wadge-degree θ_α (that is iff $\theta_\alpha < \Theta$) otherwise $\alpha = \beta$;
- $\theta_\lambda = \sup_{\alpha<\lambda} \theta_\alpha$ if λ is a limit ordinal.

The length of this so called Solovay sequence is a natural degree of consistency strength for models of determinancy. At least at low levels every θ corresponds to a strong cardinal below a limit of Woodin cardinals:

THEOREM 3 (Woodin,Steel). (a) *The following theories are equiconsistent:*
   - "$\mathrm{ZF} + \mathrm{AD}$";
   - "$\mathrm{ZFC} + \exists\lambda : \lambda$ *is a limit of Woodin cardinals*".

(b) *The following theories are equiconsistent:*
   - "$\mathrm{ZF} + \mathrm{AD} + \Theta > \theta_0$";
   - "$\mathrm{ZFC} + \exists\kappa, \lambda : \lambda$ *is a limit of Woodin cardinals* , κ *is* $<\lambda$*-strong*".

(c) *The following theories are equiconsistent:*
   - "$\mathrm{ZF} + \mathrm{AD}_\mathbb{R}$";
   - "$\mathrm{ZFC} + \exists\lambda : \lambda$ *is a limit of Woodin cardinals and* $<\lambda$*-strong cardinals*".

("Left" to "Right" in (c) due to Steel.)

One notices that compared to large cardinals this gives a rather coarse hierarchy. AD by itself is equiconsistent with infinitely many Woodin cardinals, but

"$\mathrm{AD}^+ + \Theta > \theta_0$" is already far stronger consistency wise than a proper class of Woodin cardinals. Determinacy theories strictly in between have so far not been extensively studied. (But see [12])

The upper bounds in the above theorem were established using the following theorem due to Woodin:

THEOREM 4 (Woodin). *Let λ be a limit of Woodin cardinals. Let $g \subset \mathrm{Col}(\omega, <\lambda)$ be generic. Let*

$$\mathbb{R}^* = \bigcup_{\alpha<\lambda} \mathbb{R}^{V[g \cap \mathrm{Col}(\omega,<\alpha)]}$$

and let Γ be the set of $A^* \subseteq \mathbb{R}^*$ such that there exists $\alpha < \lambda$ and A with

$$V\left[g \cap \mathrm{Col}(\omega, <\alpha)\right] \models A \text{ is } {<}\lambda \text{ u.B.}$$

and A^* is A's canonical extension onto $\mathbb{R}^*$.

Then $L(\Gamma, \mathbb{R}^*) \models \mathrm{AD}^+$.

(This is a simplified form. For some background on the derived model theorem see [9], useful is also [2].)

The above model is usually referred to as the derived model at λ, hence why the theorem is referred to as the "derived model theorem". Of course, the model really depends on the choice of the generic, but we shall soon see how to eliminate such ambiguity.

We will follow [3] for our theory of premice and iteration trees. When $\mathcal{T}$ is a normal tree on some premouse M we write:

- $\mathcal{M}^{\mathcal{T}}_\alpha$ for the α-th model of $\mathcal{T}$;
- $E^{\mathcal{T}}_\alpha$ the extender applied at the α-th step;
- $\leq_{\mathcal{T}}$ is the tree order, $\mathcal{D}^{\mathcal{T}}$ is the set of drops;
- for $\alpha \leq_{\mathcal{T}} \beta < \mathrm{lh}(\mathcal{T})$ such that $\mathcal{D}^{\mathcal{T}} \cap (\alpha, \beta]_{\leq_{\mathcal{T}}} = \emptyset$ we let $\iota^{\mathcal{T}}_{\alpha,\beta} : \mathcal{M}^{\mathcal{T}}_\alpha \to \mathcal{M}^{\mathcal{T}}_\beta$ be the iteration embedding.

If $\mathcal{T}$ has a last model, it will be referred to as $\mathcal{M}^{\mathcal{T}}$; if $\mathcal{D}^{\mathcal{T}} \cap [0, \mathrm{lh}(\mathcal{T}) - 1]_{\leq_{\mathcal{T}}} = \emptyset$, then we write $\iota^{\mathcal{T}}$ for $\iota^{\mathcal{T}}_{0,\mathrm{lh}(\mathcal{T})-1}$. If $\mathcal{T}$ is of limit type and b is a cofinal wellfounded branch, then we write $\mathcal{M}^{\mathcal{T}}_b$ for the induced limit model. If additionally $b \cap \mathcal{D}^{\mathcal{T}} = \emptyset$ then we write $\iota^{\mathcal{T}}_b : \mathcal{M}^{\mathcal{T}}_0 \to \mathcal{M}^{\mathcal{T}}_b$ for the direct limit embedding.

All trees appearing in this paper will be stacks of normal trees $\vec{\mathcal{T}} = \langle \mathcal{T}_\alpha : \alpha < \mathrm{lh}(\vec{\mathcal{T}}) \rangle$, that means

- $\mathcal{T}_0$ is a normal tree on M,
- $\mathcal{T}_\alpha$ has a last model for all α sucht that $\alpha + 1 < \mathrm{lh}(\vec{\mathcal{T}})$,
- $\mathcal{T}_{\alpha+1}$ is a normal tree on $\mathcal{M}^{\mathcal{T}_\alpha}$ for all $\alpha + 1 < \mathrm{lh}(\vec{\mathcal{T}})$,
- $\iota^{\mathcal{T}_\alpha}$ exists for all but finitely many $\alpha < \mathrm{lh}(\vec{\mathcal{T}})$,
- $\mathcal{T}_\lambda$ is a normal tree on the direct limit of $\langle \mathcal{M}^{\mathcal{T}_\alpha}, \iota^{\mathcal{T}_\alpha} : \alpha < \lambda \rangle$ whenever $\lambda < \mathrm{lh}(\vec{\mathcal{T}})$ is a limit ordinal.

For $\alpha \leq \beta < \mathrm{lh}(\vec{\mathcal{T}})$ we will write $\iota^{\vec{\mathcal{T}}}_{\alpha,\beta} : \mathcal{M}^{\mathcal{T}_\alpha}_0 \to \mathcal{M}^{\mathcal{T}_\beta}_0$ for the iteration embeddings if they exist.

Now let M be a countable, $(\omega_1 + 1)$-iterable premouse and let $\lambda \in M$ be a limit of Woodin cardinals of M. (In this paper we will usually want to assume that the M-cofinality of λ is not the critical point of a total extender on the

M-sequence.) The derived model of M at λ becomes more interesting due to the following remarkable theorem due to Woodin (, a proof of which can be found in [10]).

THEOREM 5 (Genericity Iteration). *Let M be a countable premouse. Let Σ be an (ω_1+1)-iteration strategy for M. Let δ be a Woodin cardinal of M and $\kappa < \delta$. Let x be any real. Then there exists a nowhere-dropping iteration tree $\mathcal{T}$ on M above κ with last model N and x is generic over N for a $\iota^{\mathcal{T}}(\delta)$-c.c., size $\iota^{\mathcal{T}}(\delta)$ (in N) forcing notion.*

This is one part of almost any proof of determinacy, but it is only complete with a notion of "capturing" a set of reals.

DEFINITION 6. Let M be a premouse, hybrid-premouse or HOD-premouse (for an introduction to the latter two concepts, see section 2) with an (ω_1+1)-iteration strategy Σ. Let $A \subseteq \mathbb{R}$ and $\delta \in M$. We say (M, Σ) captures A at δ if there is a $\mathrm{Col}(\omega, \delta)$-name τ such that for all non-dropping iteration trees $\mathcal{T}$ by Σ with last model $N := \mathcal{M}^{\mathcal{T}}$, and all $g \subset \mathrm{Col}(\omega, \iota^{\mathcal{T}}(\delta))$ generic over N

$$\iota^{\mathcal{T}}(\tau)_g = A \cap N[g].$$

We say (M, Σ) Suslin-captures A if and only if τ is of the form $p\left[\check{T}\right]$ (, that is τ is a canonical name for the projection of T in the generic extension,) for some tree T on $\omega \times \alpha$. We say (M, Σ) Suslin, co-Suslin-captures A if and only if it Suslin-captures both A and $\mathbb{R}\backslash A$.

Let M be as above, let λ be a limit of M-Woodin cardinals. Fix an (ω_1+1)-iteration strategy Σ. Let $T^{\Sigma}(M, \lambda)$ be the tree consisting of nodes of the form $(\langle \mathcal{T}_i : i < n\rangle, \langle \delta_i : i < n\rangle, \langle g_i : i < n\rangle)$ such that

- $(\langle \mathcal{T}_i : i < n\rangle, \langle \delta_i : i < n\rangle, \langle g_i : i < n\rangle) \in V$;
- $\forall i < n : \mathcal{T}_i$ is a no-where dropping ($\mathcal{D}^{\mathcal{T}_i} = \emptyset$) iteration tree on $\mathcal{M}^{\mathcal{T}_{i-1}}$ above $\iota^{\mathcal{T}_{i-1}}(\delta_{i-1})$ and below δ_i according to the $\mathcal{T}_{i-1}$-tail of Σ ($\mathcal{M}^{\mathcal{T}_{-1}} := M, \delta_{-1} := 0$),
- $\forall i < n : \mathcal{M}^{\mathcal{T}_{i-1}} \models \delta_i$ is a Woodin cardinal,
- $\forall i < n : g_i \subset \mathrm{Col}(\omega, \iota^{\mathcal{T}_i}(\delta_i))$ is generic over $\mathcal{M}^{\mathcal{T}_i}, g_i$ end-extends $\prod_{j<i} g_j$.

Usually, we can assume that M is ω-projecting and sound, and hence has a unique iteration strategy. So, from now on we will drop the superscript Σ.

Assume temporarily that we are in $V^{\mathrm{Col}(\omega,\mathbb{R})}$. Fix a complete listing $\langle x_n : n < \omega\rangle$ of V-reals.

Let $b := (\langle \mathcal{T}_i : i < \omega\rangle, \langle \delta_i : i < \omega\rangle, \langle g_i : i < \omega\rangle)$ be a cofinal branch through $T(M, \lambda)$ such that $x_i \in \mathcal{M}^{\mathcal{T}_i}[g_i]$ for all $i < \omega$. Set $\mathcal{T}_b := \bigoplus_{i<\omega} \mathcal{T}_i$, $M_b := \mathrm{dirlim}_{i<\omega}\langle \mathcal{M}^{\mathcal{T}_i}, \iota^{\mathcal{T}_i}\rangle$ (an absoluteness argument shows that this is well-founded!), ι_b the direct limit embedding, $\lambda_b := \iota_b(\lambda)$ and $g_b := \bigcup_{i<\omega} g_i$ (with some additional bookkeeping this can be assumed to be generic over M_b for $\mathrm{Col}(\omega, {<}\lambda_b)$).

We will write $\Gamma_b^{M,\lambda}$ for the point class given by the derived model theorem at λ_b (relative to g_b). Note that $\mathbb{R} \cap V = \mathbb{R} \cap L(\Gamma_b^{M,\lambda}, \mathbb{R}^V)$. We would like to have that $\Gamma_b^{M,\lambda}$ does not depend on b. This itself might be unrealistic, but we stand a better chance if we restrict ourselves to generic branches. ($T(M,\lambda) \in V$ can be understood as a forcing notion, any V-generic $G \subset T(M,\lambda)$ can be easily seen to be equivalent to a branch b_G through $T(M,\lambda)$.)

Let M be a countable premouse. An (ω_1+1)-iteration strategy Σ is good if and only if for any normal tree $\mathcal{T}$ on M with last model $\mathcal{M}^{\mathcal{T}}$ and any $\beta < \mathrm{lh}(\mathcal{T})$ the phalanx $\langle \mathcal{M}^{\mathcal{T}}_\alpha, \nu(E^{\mathcal{T}}_\alpha) : \alpha < \beta\rangle^\frown \mathcal{M}^{\mathcal{T}}$ is (ω_1+1)-iterable.

LEMMA 7 (Steel). *Let M be a premouse that is ω-projecting and sound. Let $\lambda \in M$ be a limit of M-Woodin cardinals such that $\mathrm{cof}^M(\lambda)$ is not the critical point of a total extender on the M-sequence. Assume M has a good iteration strategy. For any generic branch b through $T(M,\lambda)$ we have that $\Gamma^{M,\lambda}_b \subset V$.*

PROOF. Steel shows that any set $A \in \Gamma^{M,\lambda}_b$ is Suslin-co-Suslin captured over some iterate of M. See [7] Propositon 3.0.1 for details. ⊣

In the following, given an active premouse $(M;\in,\vec{E},F)$ we will write $M^- := (M;\in,\vec{E}) = M||(\mathrm{On}\cap M)$, that is the *reduct* of M.

DEFINITION 8. We say an active (ω_1+1)-iterable countable premouse $(M;\in,\vec{E},F)$ is a sharp mouse if and only if there exists a first order sentence φ such that $M||\mathrm{crit}(F) \models \varphi$ but for no $\alpha \in \mathrm{dom}(\vec{E})$ we have $M||\mathrm{crit}((\vec{E})_\alpha) \models \varphi$.

Now let $(M;\in,\vec{E},F)$ be a sharp mouse, let $a \in M$. We will write $\mathbb{F}^M_\kappa$ for the final model of the fully backgrounded construction of M^- using only extenders from $\vec{E}$ whose critical points are greater than κ (see [3]). $\mathbb{F}^M_\kappa$ is a (ω_1+1)-iterable passive premouse.

$\mathbb{F}^M_\kappa$ has some other remarkable properties:

- if $\delta > \kappa$ is a Woodin cardinal in M, then it is Woodin in $\mathbb{F}^M_\kappa$(, see [3] Chapter 11 ff.);
- if $\delta > \kappa$ is a Woodin cardinal in M and $N \in M||\kappa$ is in M a $(\delta+1)$-iterable premouse, then in the co-iteration of N and $\mathbb{F}^M_\kappa$ the latter wins the co-iteration and does not move in the process(, see [4] Chapter 2.3);
- if $\delta > \kappa$ is a Woodin cardinal in M and $x \in V^M_\delta$, then x is generic over $\mathbb{F}^M_\kappa$ for the δ-generator version of the extender algebra (, this is a consequence of the coherence of M's extender sequence).

Unfortunately, we do not know if whenever $\mu < \lambda$ are limit cardinals in M and μ is $<\lambda$-strong in M that μ is also $<\lambda$-strong in $\mathbb{F}^M_\kappa$ where $\kappa < \mu$. For this reason we are forced to move to an alternate construction.

We will write $\mathbb{L}^M_\kappa$ for the final model of the maximal +1-certified ms-array, as defined in [8], using only extenders from $\vec{E}$ whose critical points are greater than κ. $\mathbb{L}^M_\kappa$ is an (ω_1+1)-iterable passive premouse.

The core property of this construction is that whenever λ is a limit cardinal in M then the output of the maximal +1-certified ms-array of $M|\lambda$ using extenders with critical point larger than κ is an initial segment of $\mathbb{L}^M_\kappa$. One significant drawback is that backgrounds for extenders on the $\mathbb{L}^M_\kappa$-sequence can be partial over M. Fortunately, this is only the case for partial over $\mathbb{L}^M_\kappa$ extenders. Here, we will exclusively be interested in iterations that use only total extenders. Therefore this limitation will not affect us.

We will also consider constructions that are relativized to sets $a \in M$. In that case we will write $\mathbb{F}^M_\kappa(a)$ or $\mathbb{L}^M_\kappa(a)$ as appropriate. As a general rule of relativized premice, we require that all extenders on the sequence have critical points larger than the rank of a in addition to any requirement induced by the choice of κ.

DEFINITION 9 (Steel). A sharp mouse M rebuilds itself below λ iff for every non-dropping iteration tree $\mathcal{T}$ on M with last model $N := \mathcal{M}^\mathcal{T}$ and every $\kappa < \iota^\mathcal{T}(\lambda)$, there exists an active premouse P and a Σ_1-elementary embedding $\sigma : M \to P$ such that $P^- = \mathbb{L}^N_\kappa$.

Examples of mice that rebuild themselves include, but are not limited to:

- $M^\#_\omega$ below its unique limit of Woodin cardinals;
- the unique sound sharp mouse for the statement "there exists λ that is a limit of both Woodin cardinals and $<\lambda$-strong cardinals" below its largest limit of Woodin cardinals;
- $M^\#_{wlim}$ the unique sound sharp mouse for the statement "there exists a Woodin cardinal λ that is a limit of Woodin cardinals" below its largest limit of Woodin cardinals.

There is one very noteworthy example of a mouse that *does not* rebuild itself below its largest limit of Woodin cardinals, that is $M^\#_{refl}$ the unique sound sharp mouse for the statement "there exists λ that is a limit of Woodin cardinals and $<\lambda$-strong cardinals and there exists $\kappa < \lambda$ that reflects the set of $<\lambda$-strong cardinals below λ".

DEFINITION 10. (M, λ) is a tractable pair if and only if M is a sound countable premouse and $\lambda < \mathrm{On} \cap M$ is such that

(a) M is a sharp mouse with a good $(\omega_1 + 1)$-iteration strategy,
(b) λ is a limit of Woodin cardinals of M and its cofinality in M is not the critical point of a total extender on the M-sequence,
(c) M rebuilds itself below λ.

LEMMA 11 (Steel). *Let (M, λ) be a tractable pair. Then $\Gamma(M, \lambda)_b$ is independent of the generic branch b through $T(M, \lambda)$.*

PROOF. See [7] Lemma 3.7. ⊣

By standard forcing theory we will have that $\Gamma(M, \lambda)_b \in V$ for any given generic branch b. We will then write $D(M, \lambda) := L(\Gamma(M, \lambda)_b, \mathbb{R})$, the derived model of M at λ.

§2. HOD-**Mice.** This will be a short review of the notions and terms and the associated background knowledge we will lean on heavily during the proof. An in-depth treatise on the subject of HOD-mice (below "$AD_\mathbb{R} + \Theta$ regular", which is sufficient for our needs,) can be found in [4].

Let N be a premouse and Σ be a partial iteration strategy. A Σ-hybrid premouse M is a J-structure with N at the bottom and predicates coding an extender-sequence and $\Sigma \restriction M$. The exact way this is achieved shall not concern us here (see [6] for details).

A HOD-premouse $\mathcal{P}$ is a ZFC^--structure of the following form:

$\mathcal{P}$ is divided into layers; let $\langle \delta_i^{\mathcal{P}} : i \leq \lambda^{\mathcal{P}} \rangle$ be a complete, increasing listing of all $\mathcal{P}$-cardinals which are Woodin cardinals or limits of Woodin cardinals (inside $\mathcal{P}$), there are exactly $\lambda^{\mathcal{P}}+1$ many layers enumerated as $\langle \mathcal{P}(i) : i \leq \lambda^{\mathcal{P}} \rangle$; we have $\mathcal{P}(i) := \mathcal{P}||((\delta_i^+)^\omega)^{\mathcal{P}(i)}$ for $i \leq \lambda^{\mathcal{P}}$; we set $\mathcal{P}(i)^- := \mathcal{P}(i')$ iff $i = i'+1$, $\mathcal{P}(i)^- := \mathcal{P} \cap V_{\delta_i}^{\mathcal{P}}$ if i is a limit ordinal, and $\mathcal{P}(0)^- := \emptyset$; $\mathcal{P}(i)$ is a hybrid mice over $\mathcal{P}(i)^-$ for all $i \leq \lambda^{\mathcal{P}}$; we require that $((\delta_i^{\mathcal{P}})^+)^{\mathcal{P}(i)} = ((\delta_i^{\mathcal{P}})^+)^{\mathcal{P}}$ for all limit ordinals less than $\lambda^{\mathcal{P}}$(, note that this is false for successor ordinals); finally, if $i \leq \lambda^{\mathcal{P}}$ is a successor or 0 and $\eta < \delta_i^{\mathcal{P}}$ then $\mathcal{P}||\eta$ has an $(\mathrm{On}^{\mathcal{P}(i)}, \mathrm{On}^{\mathcal{P}(i)})$-iteration strategy definable over $\mathcal{P}(i)$.

DEFINITION 12. A HOD-pair $(\mathcal{P}, \Sigma)$ is a pair such that $\mathcal{P}$ is a countable HOD-premouse and Σ is an (ω_1, ω_1)-iteration strategy such that $\Sigma_{\mathcal{Q}(\alpha),\mathcal{T}} \restriction \mathcal{Q} = \Sigma_\alpha^{\mathcal{Q}}$ for all non-dropping (, that is $\iota^{\mathcal{T}}$ exists,) iteration trees $\mathcal{T}$ according to Σ with last model $\mathcal{Q}$ and all $\alpha < \lambda^{\mathcal{Q}}$. Here $\Sigma_{\mathcal{Q}(\alpha),\mathcal{T}}$ refers to the iteration strategy on $\mathcal{Q}(\alpha)$ induced by Σ and $\Sigma_\alpha^{\mathcal{Q}}$ is the partial iteration strategy on the $\mathcal{Q}$-sequence.

Remark 13. Come section 5 we will want to allow for HOD-pairs to have $\mathcal{P}$ be uncountable. In that case Σ will be at least a $(\mathrm{Card}(\mathcal{P})^+, \mathrm{Card}(\mathcal{P})^+)$-iteration strategy and the expectation is that Σ naturally extends to a (ω_1, ω_1)-iteration strategy Σ^* over $V^{\mathrm{Col}(\omega,\mathcal{P})}$ and that, in fact, Σ^* is contained in some determinacy model.

For the remainder of this section and for the remainder of this section only our background theory will be $\mathrm{ZF}+\mathrm{AD}^+$.

We say a strategy Σ has *hull condensation* if and only if for all trees $\mathcal{T}$ by Σ and $\bar{\mathcal{T}}$ a *hull* of $\mathcal{T}$, we have that $\bar{\mathcal{T}}$ is also by Σ(, for the precise definition of hull see [4] Chapter 1.6).

Let us fix a pointclass Γ. For an (ω_1, ω_1)-iteration strategy $\Sigma \in \Gamma$ (for some fixed coding of Σ) with hull condensation and a a set we write $Lp^{\Gamma,\Sigma}(a)$ for the union of all sound-above-a hybrid premice over a all of whose countable hulls are Σ-hybrid premice with (ω_1, ω_1)-iteration strategies coded by sets in Γ.

DEFINITION 14. Let Γ be a pointclass. $(\mathcal{P}, \Sigma)$ is a Γ HOD-pair if it is a HOD-pair and for all iteration trees $\mathcal{T}$ on $\mathcal{P}$ according to Σ with last model $\mathcal{Q}$ we have $\Sigma_{\mathcal{Q}(\alpha)^-,\mathcal{T}} \in \Gamma$ for all $\alpha \leq \lambda^{\mathcal{Q}}$ and we have for all cutpoints $\eta \in \mathcal{Q}$ that $\mathcal{Q}||(\eta^+)^{\mathcal{Q}(\alpha+1)} \trianglelefteq \mathrm{Lp}^{\Gamma,\Sigma_{\mathcal{Q}(\alpha),\mathcal{T}}}(\mathcal{Q}||\eta)$ for $\alpha \leq \lambda^{\mathcal{Q}}$ minimal with $\eta \in \mathcal{Q}(\alpha)$.

DEFINITION 15. Let $\mathcal{P}$ and $\mathcal{Q}$ be HOD-premice. We say $\mathcal{Q}$ is a HOD-initial segment of $\mathcal{P}$, written $\mathcal{Q} \trianglelefteq_{\mathrm{HOD}} \mathcal{P}$, iff there exists $\alpha \leq \lambda^{\mathcal{P}}$ such that $\mathcal{Q} = P(\alpha)$. We shall also sometimes consider $\mathcal{P}^-$ to be a HOD-initial segment of $\mathcal{P}$.

In the following we will have to deal with "improper" limit type HOD-pairs including but not strictly limited to the anomalous type III HOD-pairs of [4] Chapter 3.4. A typical situation is as follows: let Γ be a pointclass and $\langle \langle \mathcal{P}_\alpha, \Sigma_\alpha \rangle : \alpha < \eta \rangle$ a $\trianglelefteq_{\mathrm{HOD}}$-increasing sequence of Γ HOD-pairs such that the restriction of Σ_α to trees on $\mathcal{P}_\beta$ is Σ_β whenever $\beta \leq \alpha < \eta$.

Let $\Sigma := \bigoplus_{\alpha<\eta} \Sigma_\alpha$ and $\mathcal{P} := \bigcup_{\alpha<\eta} \mathcal{P}_\alpha$. Let then $\mathcal{P}^+$ be a possibly proper initial segment of $\mathrm{Lp}^{\Gamma,\Sigma}(\mathcal{P})$ end-extending $\mathcal{P}$. We will let $\lambda^{\mathcal{P}} := \sup_{\alpha<\eta} \lambda^{\mathcal{P}_\alpha}$.

There are two main reasons Σ can fail to be an iteration strategy for $\mathcal{P}^+$: the more immediate is that a given tree on some $\mathcal{P}_\alpha$ by Σ_α could produce ill-founded models taken as a tree on $\mathcal{P}^+$; the less immediate but more severe reason is that if $\mathrm{cof}^{\mathcal{P}^+}(\lambda^{\mathcal{P}})$ is the critical point of a total over $\mathcal{P}^+$ extender then iterations on $\mathcal{P}^+$ will generate new layers of our HOD-premouse above the pointwise image of $\lambda^{\mathcal{P}}$, Σ does not contain any information on how to iterate these new layers.

The problem is then in defining an extension of Σ that does. The proof of our main theorem will introduce a way of doing just that.

Definition 16. Let $(\mathcal{P}, \Sigma)$ be a Γ HOD-pair. Σ is Γ-fullness preserving if and only if for all $\mathcal{T}$ according to Σ with last model $\mathcal{Q} := \mathcal{M}^{\mathcal{T}}_\gamma$ such that $[0,\gamma]_{\mathcal{T}}$ does not drop, $\mathrm{Lp}^{\Gamma,\Sigma_{\mathcal{Q}(\alpha)^-,\mathcal{T}}}(\mathcal{Q}||\beta) \subset \mathcal{Q}$ for all cutpoints β of $\mathcal{Q}$ and α minimal with $\beta \in \mathcal{Q}(\alpha)$.

Definition 17. Let M be a premouse, hybrid-premouse or HOD-premouse. Let Σ be an iteration strategy. We say Σ has branch condensation if and only if for all iteration trees $\mathcal{U}$ of limit type according to Σ, and all branches b through $\mathcal{U}$, if there exists a non-dropping Σ-iterate N, say $j : M \to N$ is the iteration embedding, and an elementary embedding $\pi : \mathcal{M}^{\mathcal{U}} \to N$ such that $\pi \circ \iota^{\mathcal{U}}_b = j$ then $b = \Sigma(\mathcal{U})$.

In general, we cannot expect that any given pair $((\mathcal{P}, \Sigma), (\mathcal{Q}, \Lambda))$ of HOD-pairs can be compared. That is because $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ might be tied to different pointclasses. Fortunately, this proves to be the only serious limitation.

Theorem 18 (Sargsyan). *Assume that there is no inner model $M \subsetneq L(\mathcal{P}(\mathbb{R}))$ containing all the reals that satisfies "$\mathrm{AD}_{\mathbb{R}} + \Theta$ is regular". Let Γ be a constructively closed pointclass ($\Gamma = \mathcal{P}(\mathbb{R}) \cap L(\Gamma, \mathbb{R})$), and let $((\mathcal{P}, \Sigma), (\mathcal{Q}, \Lambda))$ be a pair of Γ HOD-pairs such that both Σ and Λ have branch condensation and are Γ-fullness preserving.*

Then there exists non-dropping trees $\mathcal{T}$ on $\mathcal{P}$ by Σ with last model $\mathcal{P}^*$ and $\mathcal{U}$ on $\mathcal{Q}$ by Λ with last model $\mathcal{Q}^*$ such that
- *$\mathcal{Q}^* \trianglelefteq_{\mathrm{HOD}} \mathcal{P}^*$ and $\Sigma_{\mathcal{P}^*,\mathcal{T}} \restriction \mathcal{Q}^* = \Lambda_{\mathcal{Q}^*,\mathcal{U}}$ or*
- *$\mathcal{P}^* \trianglelefteq_{\mathrm{HOD}} \mathcal{Q}^*$ and $\Lambda_{\mathcal{Q}^*,\mathcal{U}} \restriction \mathcal{P}^* = \Sigma_{\mathcal{P}^*,\mathcal{T}}$.*

We call $(\mathcal{P}^*, \Sigma_{\mathcal{P}^*,\mathcal{T}})$ a tail of $(\mathcal{P}, \Sigma)$.

The next theorem states that the HOD of a determinacy model is, in essence, a HOD-premouse. Not only does this motivate our interest in HOD-mice, but it also proves that not all terms of inner model theory are the result of a flight of fancy.

Theorem 19 (Woodin-Steel,Sargsyan). *Assume that there is no inner model $M \subsetneq L(\mathcal{P}(\mathbb{R}))$ containing all the reals that satisfies "$\mathrm{AD}_{\mathbb{R}} + \Theta$ is regular". Let $(\mathcal{P}, \Sigma)$ be a HOD-pair such that Σ is $\mathcal{P}(\mathbb{R})$-fullness preserving and has branch condensation. Let*

$$\mathcal{D} := \langle (\mathcal{Q}, \Lambda), \pi_{\mathcal{Q},\mathcal{R}} : (\mathcal{Q}, \Lambda) \leq_{\mathcal{D}} (\mathcal{R}, \Psi) \rangle$$

be the directed system of all Σ-iterates of $(\mathcal{P}, \Sigma)$ together with the iteration embeddings, that is its domain consists of all non-dropping Σ-tails of $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda) \leq_{\mathcal{D}} (\mathcal{R}, \Psi)$ iff $(\mathcal{R}, \Psi)$ is an Λ-iterate of $(\mathcal{Q}, \Lambda)$. (It is an elementary fact *about* HOD*-pairs as above that given* $\mathcal{Q}, \mathcal{R} \in \mathcal{D}$*,* $\mathcal{R}$ *an iterate of* $\mathcal{Q}$*, the iteration embedding does not depend on the specific tree used to go from* $\mathcal{Q}$ *to* $\mathcal{R}$*.)*

Let $\mathcal{H}$ *be the direct limit.* $\pi^{\Sigma}_{\mathcal{Q},\infty} : \mathcal{Q} \to \mathcal{H}$ *the direct limit embedding. Then*

$$(HOD \cap V_{\theta_{\pi^{\Sigma}_{\mathcal{Q},\infty}(\alpha)}}) = |(\mathcal{H}||\pi^{\Sigma}_{\mathcal{Q},\infty}(\delta^{\mathcal{Q}}_{\alpha}))|$$

for all $(\mathcal{Q}, \Lambda) \in \mathcal{D}$ *and all* $\alpha \leq \lambda^{\mathcal{Q}}$*.*

This theorem has a converse which essentially follows from the "Generation of Pointclasses" theorem of [4].

THEOREM 20 (Sargsyan). *Assume that there is no inner model* $M \subsetneq L(\mathcal{P}(\mathbb{R}))$ *containing all the reals that satisfies "*$\mathrm{AD}_{\mathbb{R}} + \Theta$ *is regular". Let* $\theta_\alpha < \Theta$*. Then there exists a* HOD*-pair* $(\mathcal{P}, \Sigma)$ *such that* Σ *is* $\mathcal{P}(\mathbb{R})$*-fullness preserving and has branch condensation and* $\pi^{\Sigma}_{\mathcal{P},\infty}(\lambda^{\mathcal{P}}) = \alpha$*.*

The two theorems form the basis for "HOD-analysis". (Strictly speaking, it only tells us how to compute the HOD of models of $\mathrm{AD}_{\mathbb{R}}$. We can and will ignore models here that do not satisfy $\mathrm{AD}_{\mathbb{R}}$.)

Unfortunately, there is no known proof of HOD-analysis without some smallness assumptions. All known proofs follow the same basic template as seen in [11]. The crux being that the proof depends on "mouse capturing" which we do not know how to prove in general. Sargsyan has shown that it holds in the minimal model of "$\mathrm{AD}_{\mathbb{R}} + \Theta$ regular" and below. (Recent work has extended this to minimal models of the axiom LSA, see [5].)

THEOREM 21 (Sargsyan). *Assume that there is no inner model* $M \subsetneq L(\mathcal{P}(\mathbb{R}))$ *containing all the reals that satisfies "*$\mathrm{AD}_{\mathbb{R}} + \Theta$ *is regular". Let* $(\mathcal{P}, \Sigma)$ *be a* HOD*-pair such that* Σ *is* $\mathcal{P}(\mathbb{R})$*-fullness preserving and has branch condensation. Let* $x, y \in \mathbb{R}$ *be such that* y *is ordinal definable from* x *and* Σ*. Then* $y \in \mathrm{Lp}^{\Sigma}(x)$*.*

What we need to realize is that given a HOD-pair $(\mathcal{P}, \Sigma)$ if $\pi^{\Sigma}_{\mathcal{P},\infty}(\lambda^{\mathcal{P}}) > \beta$, then $\Theta > \theta_\beta$. This is what will allow us to prove that certain mice's derived model have sufficiently long Solovay sequences.

A problem we need to adress is that a given HOD-premouse will usually not have a unique iteration strategy. We will need to fix parameters then from which an iteration strategy can be defined. For this it is important that one stays within a given hierarchy. If Σ is an iteration strategy for some fine-structural model and M is a Σ-premouse, any iteration on M shall be understood as being above N.

Q-structures are one possible way to uniquely pick branches. Let M be a (hybrid)-premouse. Let $\mathcal{T}$ be an iteration tree on M and let b be a cofinal branch through $\mathcal{T}$. $Q(b, \mathcal{T})$ (if it exists) is the least initial segment Q of the well-founded part of $\mathcal{M}^{\mathcal{T}}_b$ such that Q defines a failure of $\delta(\mathcal{T})$ to be Woodin or $\rho_\omega(Q) < \delta(\mathcal{T})$, and the phalanx $\Phi(\mathcal{T})^\frown Q$ is countably iterable. (Remember $\delta(\mathcal{T}) = \sup_{\alpha < \mathrm{lh}(\mathcal{T})} \mathrm{lh}(E^{\mathcal{T}}_\alpha)$ and $\mathcal{M}(\mathcal{T}) = \bigcup_{\alpha < \mathrm{lh}(\mathcal{T})} (M^{\mathcal{T}}_\alpha || \mathrm{lh}(E^{\mathcal{T}}_\alpha))$.)

LEMMA 22 (Martin-Steel). *Let* M *be a (hybrid)-premouse. Let* $\mathcal{T}$ *be an iteration strategy on* M *and let* b *and* c *be distinct cofinal branches through* $\mathcal{T}$*.*

- $\mathrm{ran}(i^{\mathcal{T}}_b) \cap \mathrm{ran}(i^{\mathcal{T}}_c)$ *is bounded below* $\delta(\mathcal{T})$*;*

- $\delta(\mathcal{T})$ *is Woodin relative to sets in* $\mathrm{wfp}(M_b^{\mathcal{T}}) \cap \mathrm{wfp}(M_c^{\mathcal{T}})$.

This is the famous "zipper" argument, see [10]. As a consequence we have:

COROLLARY 23. *Let M be a (hybrid)-premouse. Let $\mathcal{T}$ be an iteration strategy on M and let b and c be distinct cofinal branches through $\mathcal{T}$. Then at most one of $Q(b,\mathcal{T})$ and $Q(c,\mathcal{T})$ exists.*

Let now Γ be a pointclass and let $(\mathcal{P},\Sigma)$ be a Γ HOD-pair such that Σ is Γ-fullness preserving and has branch condensation. Assume $\lambda^{\mathcal{P}} = \alpha+1$. Let $\mathcal{T}$ be a normal iteration tree on $\mathcal{P}$ above $\delta_\alpha^{\mathcal{P}}$ of limit type. There are three cases:

(a) $\mathcal{T}$ has a fatal drop, that is there exists some $\beta < \mathrm{lh}(\mathcal{T})$ such that $\mathcal{T}_{\geq\beta}$ can be considered an iteration on $M_\beta^{\mathcal{T}}$ above some γ such that $\rho_\omega(M_\beta^{\mathcal{T}}) \leq \gamma$ and $M_\beta^{\mathcal{T}} \trianglelefteq \mathrm{Lp}^{\Gamma,\Sigma_{\mathcal{P}(\alpha)}}(\mathcal{M}_\beta^{\mathcal{T}}||\gamma)$ (, in that case $\mathcal{T}_{\geq\beta}$ is uniquely determined);
(b) there exists $Q \trianglelefteq \mathrm{Lp}^{\Gamma,\Sigma_{\mathcal{P}(\alpha)}}(\mathcal{M}(\mathcal{T}))$ such that Q defines a failure of $\delta(\mathcal{T})$ to be Woodin or $\rho_\omega(Q) < \delta(\mathcal{T})$, and $\Sigma(\mathcal{T})$ is the unique cofinal well-founded branch b such that $Q = Q(b,\mathcal{T})$;
(c) $\mathrm{Lp}^{\Gamma,\Sigma_{\mathcal{P}(\alpha)}}(\mathcal{M}(\mathcal{T})) \models$ "$\delta(\mathcal{T})$ is a Woodin cardinal" and letting $b := \Sigma(\mathcal{T})$ we have $i_b^{\mathcal{T}}(\delta_{\alpha+1}^{\mathcal{P}}) = \delta(\mathcal{T})$ and $M_b^{\mathcal{T}} = \mathrm{Lp}_\omega^{\Gamma,\Sigma_{\mathcal{P}(\alpha)}}(\mathcal{M}(\mathcal{T}))$.

In cases (a) and (b) we call $\mathcal{T}$ *short* and then, naturally, in case (c) we call them *maximal*. Note that while $b := \Sigma(\mathcal{T})$ is not the only branch with the required property for maximal $\mathcal{T}$, it is uniquely determined by $\mathrm{ran}(i_b^{\mathcal{T}}) \cap \delta(\mathcal{T})$ or any set cofinal in it by Lemma 22 (a).

Two things are of note here: firstly, here we consider $\mathcal{P}$ as essentially a $\Sigma_{\mathcal{P}^-}$-hybrid mouse; to fully determine a given iteration strategy one needs to apply this process in stages, facilitated by a book keeping device we shall introduce shortly; secondly, $\mathrm{Lp}_\omega^{\Gamma,\Sigma_{\mathcal{P}(\alpha)}}(\cdot)$ refers to the ω-th iteration of the $\mathrm{Lp}^{\Gamma,\Sigma_\alpha^{\mathcal{P}}}(\cdot)$ function; this is not quite standard but it improves readability in our situation.

We should also mention that any structure closed under the $\mathrm{Lp}^{\Gamma,\Sigma_{\mathcal{P}(\alpha)}}$-operator can track most of the iteration. For a maximal tree $\mathcal{T}$ the structure might not be able to identify its branch, but it can identify the target model of the branch embedding. As $\delta(\mathcal{T})$ is a cutpoint of the target model, a normal iteration such as a comparison or genericity iteration will end there, this can be very useful. This tracking process is referred to as a "pseudo-iteration". ([5] introduces the term "short-tree strategy" which is based on the same concept.)

We end this section by introducing the aforementioned book keeping device.

DEFINITION 24 (Essential components). Let $\mathcal{P}$ be a hod premouse, $\lambda^{\mathcal{P}}$ a limit ordinal and let $\vec{\mathcal{T}}$ be a stack of normal iteration trees on $\mathcal{P}$. We call a sequence $\langle \mathcal{M}_\alpha, \mathcal{M}_\alpha^*, \vec{\mathcal{T}}_\alpha, i_{\alpha,\beta} : \alpha \leq \beta < \eta\rangle$ the essential components of $\vec{\mathcal{T}}$ iff:

- $\mathcal{M}_0 = \mathcal{P}$, $\mathcal{M}_0^*$ is the least HOD-initial segment of $\mathcal{P}$ such that the first extender used in $\vec{\mathcal{T}}$ is in $\mathcal{M}_0^*$ and $\vec{\mathcal{T}}_0$ is the largest initial segment of $\vec{\mathcal{T}}$ that is based on $\mathcal{M}_0^*$;
- for any α, if $\vec{\mathcal{T}}_\beta$ is defined for all $\beta < \alpha$, then $\bigoplus_{\beta<\alpha} \vec{\mathcal{T}}_\beta \trianglelefteq \vec{\mathcal{T}}$;
- if $\bigoplus_{\beta<\alpha} \vec{\mathcal{T}}_\beta = \vec{\mathcal{T}}$ for some α, then $\eta = \alpha$ and $\mathcal{M}_\alpha, \mathcal{M}_\alpha^*, \vec{\mathcal{T}}_\alpha$ are undefined;

- if $\bigoplus_{\beta<\alpha} \vec{\mathcal{T}}_\beta \triangleleft \vec{\mathcal{T}}$ then $\mathcal{M}_\alpha$ is the last model of $\bigoplus_{\beta<\alpha} \vec{\mathcal{T}}_\beta$ in $\vec{\mathcal{T}}$, and $\mathcal{M}^*_\alpha$ is the largest HOD-initial segment of $\mathcal{M}_\alpha$ such that the next extender used in $\vec{\mathcal{T}}$ is in $\mathcal{M}^*_\alpha$;
- if $\mathcal{M}^*_\alpha$ is defined then $\vec{\mathcal{T}}_\alpha$ is the largest initial segment of $\vec{\mathcal{T}} - \bigoplus_{\beta<\alpha} \vec{\mathcal{T}}_\beta$ that is entirely on $\mathcal{M}^*_\alpha$;
- $i_{\alpha,\beta} : \mathcal{M}_\alpha \to \mathcal{M}_\beta$ is the iteration embedding (if it exists).

§3. Hyperstrongness. Define by induction the following notion: Let $\kappa < \lambda$ be cardinals. We say κ is $(<\lambda, \alpha)$-strong if for all $\beta < \alpha$ and all $\gamma < \lambda$ there exists an extender E on κ such that $\mathrm{Ult}(V; E)$ is well-founded, $V_\gamma \subseteq \mathrm{Ult}(V; E)$, and κ remains $(<i_E(\lambda), \beta)$-strong in $\mathrm{Ult}(V; E)$. (See also [1].)

This notion is not necessarily consistent with arbitrary values of α and λ. In fact:

Lemma 25. *Let λ be a limit of measurable cardinals and not itself of measurable cofinality. Then there is some $\Delta < \lambda^+$ such that no $\kappa < \lambda$ is $(<\lambda, \Delta)$-strong.*

Proof. Assume not. For every total extender $E \in V_\lambda$ the set $C_E \subset \lambda^+$ of ordinals which are fixed by its ultrapower embedding is a λ-club. (By choice of λ any i_E will fix λ and thus λ^+.) Thus

$$C := \bigcap_{E \in V_\lambda} C_E$$

has ordertype λ^+. But for $\alpha < \beta$ both in C the least κ which is $(<\lambda, \alpha)$ strong is less than the least κ which is $(<\lambda, \beta)$-strong.

To see this, let κ be $(<\lambda, \beta)$-strong. There is then some $j : V \to M$ with critical point κ such that κ is still $(<\lambda, \alpha)$-strong in M. Because $\alpha \in C$ we then have that κ is $(<j(\lambda), j(\alpha))$-strong in M. By elementarity, there must then be some $\mu < \kappa$ that is $(<\lambda, \alpha)$-strong in V.

Therefore we get a strictly increasing sequence in λ of length λ^+. Clearly, this is absurd! $\dashv$

We do not know what exactly Δ is, but it is larger than the length of nice prewellorders on V_λ.

Let x be a set, such that $x^\#$ exists, and let λ be an uncountable cardinal such that $\mathrm{rank}(x) < \lambda$. Let I_x be the set of Silver indiscernibles for x. Write $M^\#(\alpha, x) := (J_\beta[x]; \in, F(\alpha, x))$ for the α-th iterate of $x^\#$ by its top extender. Define a prewellorder R^λ_x on pairs (n, a) where n is a Gödel number for some $r\Sigma_1$-term in the language of $x^\#$ and a is a finite set of indiscernibles less than λ.

Set $(\ulcorner\tau\urcorner, a) R^\lambda_x (\ulcorner\sigma\urcorner, b)$ iff $\tau^{M^\#(\lambda+1,x)}(x, a \cup \{\lambda\}) \leq \tau^{M^\#(\lambda+1,x)}(x, b \cup \{\lambda\})$ and both of which are less than $\mathrm{crit}(F(\lambda+1, x))$.

To save on ink and sanity we shall write Δ^λ_x for the length of R^λ_x and $\delta^\lambda_x(s)$ for the norm relative to R^λ_x of any given element s of its domain.

Remark. $\Delta^\lambda_x = \mathrm{crit}(F(\lambda+1, x))$ because $I_x \cap (\lambda+1)$ is a set of generators for $M^\#(\lambda+1, x)$.

As far as ultrapowers are concerned R^λ_x has a very strong absoluteness property: if E is some extender on V producing a well-founded ultrapower M such that $x \in M$ and $i_E(\lambda) = \lambda$ then $i_E(\delta_x^\lambda) = \delta_x^\lambda$, this is because both models share the same ordinals. In section 6 we will deal with relations which do not, as far as we know, have this property.

LEMMA 26. *Let λ be a Woodin cardinal. Then for all $x \in V_\lambda$ there exists a $\kappa < \lambda$ that is $({<}\lambda, \Delta_x^\lambda)$-strong.*

PROOF. Fix x. Define

$$A_x := \{(\mu, s) | \mu \text{ is } ({<}\lambda, \delta_x^\lambda(s))\text{-strong}\}.$$

Let $\kappa < \lambda$ be A_x-reflecting. We will prove by induction on R_x^λ that κ is $({<}\lambda, \Delta_x^\lambda)$-strong. Let s be arbitrary such that for all $rR_x^\lambda s$ we already know that κ is $({<}\lambda, \delta_x^\lambda(r))$-strong. Let $\gamma < \lambda$ and $rR_x^\lambda s$ be arbitrary. Then there exists some extender E on κ with strength bigger than γ that reflects A_x up to γ. By assumption κ is $({<}\lambda, \delta_x^\lambda(r))$-strong. So $(\kappa, r) \in A_x$. Without loss of generality $\mathrm{rank}(r) < \gamma$ thus $(\kappa, r) \in i_E(A_x)$.

Thereby κ is $({<}\lambda, \delta_x^\lambda(r))$-strong in $\mathrm{Ult}(V; E)$. (Here we use $i_E(\delta_x^\lambda)(r) = \delta_x^\lambda(r)$.) But this shows that κ is in fact $({<}\lambda, \delta_x^\lambda(s))$-strong completing the induction step! ⊣

One easily notices that we can get a κ as above that is actually $({<}\lambda, \Delta_x^\lambda)$-strong for all $x \in V_\kappa$.

With some additional work we can get the same result from the weaker assumption that there is some cardinal that reflects the set of ${<}\lambda$-strong cardinals.

LEMMA 27. *Let λ be a limit of ${<}\lambda$-strong cardinals. Let $\kappa < \lambda$ be such that κ reflects ${<}\lambda$-strong cardinals below λ. Then for all $x \in V_\kappa$, κ is $({<}\lambda, \Delta_x^\lambda)$-strong.*

PROOF. Fix $x \in V_\kappa$.

Claim 1. For all $\kappa < \alpha < \lambda$ inaccessible, if κ is $({<}\lambda, \delta_x^\lambda(s))$-strong, then it is $({<}\alpha, \delta_x^\alpha(s))$ strong, assuming $\max(s) < \alpha$.

PROOF OF CLAIM. Proof is by induction on R_x^λ: let α be as above and assume κ is $({<}\lambda, \delta_x^\lambda(s))$-strong. Let $t \in \mathrm{dom}(R_x^\alpha)$ be such that $tR_x^\alpha s$. Let $\beta < \alpha$. By assumption there exists E such that

$$\mathrm{Ult}(V; E) \models \kappa \text{ is } ({<}\lambda, \delta_x^\lambda(t))\text{-strong},$$

and $V_\beta \subset \mathrm{Ult}(V; E)$. Let $\beta < \eta < \alpha$ be inaccessible such that $t \in V_\eta$. Let $i : \mathrm{Ult}(V; E \restriction \eta) \to \mathrm{Ult}(V; E)$ be the factor embedding. Because $\mathrm{crit}(i) \geq \eta$ we have

$$\mathrm{Ult}(V; E \restriction \eta) \models \kappa \text{ is } ({<}\lambda, \delta_x^\lambda(t))\text{-strong}.$$

By our inductive assumption we can then conclude

$$\mathrm{Ult}(V; E \restriction \eta) \models \kappa \text{ is } ({<}\alpha, \delta_x^\alpha(t))\text{-strong}.$$

Of course, η can be chosen such that $V_\beta \subset \mathrm{Ult}(V; E \restriction \eta)$. Note also that α is inaccessible and $tR_x^\lambda s$ in $\mathrm{Ult}(V; E \restriction \eta)$. □

We have just shown that $({<}\lambda, \Delta_x^\lambda)$-strongness reflects downward to inaccessible cardinals. We will now show that it reflects upwards at strong cardinals.

Claim 2. For all $\kappa < \alpha < \lambda$ which are strong, if κ is $({<}\alpha, \delta_x^\alpha(s))$ strong, then it is $({<}\lambda, \delta_x^\lambda(s))$-strong.

PROOF OF CLAIM. Proof is by induction on R^λ_x: let α be as above and assume κ is $(<\alpha, \delta^\alpha_x(s))$-strong. Let $t \in \mathrm{dom}(R^\lambda_x)$ be such that $tR^\lambda_x s$.

Let $\alpha < \beta < \eta < \lambda$ be such that $t \in V_\eta$ and η is inaccessible. Because α is strong we get some $i : V \to N$ with $\mathrm{crit}(i) = \alpha$, $V_\eta \subset N$ and $i(\alpha) > \eta$. By elementarity

$$N \models \kappa \text{ is } (<i(\alpha), \delta^{i(\alpha)}_x(s))\text{-strong.}$$

So, there is some F such that $\mathrm{crit}(F) = \kappa$, $V_\beta \subseteq \mathrm{Ult}(N; F)$ and κ is $(<i(\alpha), \delta^{i(\alpha)}_x(t))$-strong in $\mathrm{Ult}(N; F)$. Here we use that $tR^{i(\alpha)}_x s$ in N.

By the previous claim we have:

$$\mathrm{Ult}(N; F) \models \kappa \text{ is } (<i_F(\alpha), \delta^{i_F(\alpha)}_x(t))\text{-strong}$$

Also $V^{\mathrm{Ult}(V;F)}_{i_F(\alpha)} = V^{\mathrm{Ult}(N;F)}_{i_F(\alpha)}$ and therefore

$$\mathrm{Ult}(V; F) \models \kappa \text{ is } (<i_F(\alpha), \delta^{i_F(\alpha)}_x(t))\text{-strong.}$$

By our inductive assumption we can then conclude

$$\mathrm{Ult}(V; F) \models \kappa \text{ is } (<i_F(\lambda), \delta^{i_F(\lambda)}_x(t))\text{-strong.}$$

□

Now we can finish the proof: assume we have already shown that κ is $(<\lambda, \delta^\lambda_x(s))$-strong for some $s \in V_\lambda$. Take some $<\lambda$-strong $\kappa < \alpha < \lambda$ with $s \in V_\alpha$. We will show that there is some $j : V \to M$ with $\mathrm{crit}(j) = \kappa$, $V_\alpha \subset M$ and κ is $(<\lambda, \delta^\lambda_x(s))$-strong in M. As $<\lambda$-strongs are cofinal in λ this will suffice.

By the first claim we have that κ is $(<\alpha, \delta^\alpha_x(s))$-strong. Take some $j : V \to M$ with $\mathrm{crit}(j) = \kappa$, $V_\alpha \subset M$ and α is $<\lambda$-strong in M. We have that κ is $(<\alpha, \delta^\alpha_x(s))$-strong in M because that is a statement about V_α only. By the second claim we then have that κ is $(<\lambda, \delta^\lambda_x(s))$-strong in M as wanted. ⊣

§4. Mice with many hyperstrongs.

DEFINITION 28. By $M^\#_{\theta_{\aleph_2}}$ we refer to the least $(\omega_1 + 1)$ -iterable (Mitchell-Steel-)premouse $(M; \in, \vec{E}, F)$ where F is non-empty and $M||\,\mathrm{crit}(F)$ satisfies the sentence:

"There is some λ, limit of Woodin cardinals, such that for all $x \in V_\lambda$ there exist some $\kappa < \lambda$ which is $(<\lambda, \Delta^\lambda_x)$-strong".

This name reflects our belief that this mouse is the least whose derived model will satisfy $\Theta = \theta_{\aleph_2}$. As promised we will also introduce a variant that will partially answer a question of John Steel from [7].

DEFINITION 29. By $M^\#_{>\theta_{\aleph_2}}$ we refer to the least $(\omega_1 + 1)$ -iterable (Mitchell-Steel-)premouse $(M; \in, \vec{E}, F)$ where F is non-empty and $M||\,\mathrm{crit}(F)$ satisfies the sentence:

"There is some λ, limit of Woodin cardinals, such that for all $\alpha < \lambda$ there exists some $\alpha < \kappa < \lambda$ which is $(<\lambda, \Delta^\lambda_x)$-strong for all $x \in V_\kappa$".

This is certainly not the least mouse whose derived model satisfies $\Theta > \theta_{\omega_2}$ but it might conceivably be the least mouse whose derived model satisfies $\Theta > \theta_{\omega_2}$ and rebuilds itself below its largest limit of Woodin cardinals.

It is now time to keep to our promise that our mice will be well-behaved in the sense of Section 1.

LEMMA 30. *Assume that $M := M^{\#}_{\theta_{\aleph_2}}$ exists and has a good (ω_1+1)-iteration strategy. Then M together with λ, its distinguished limit of Woodin cardinals and hyperstrong cardinals, form a tractable pair.*

PROOF. Clearly, M is a sharp mouse, by assumption it has a good strategy, λ's cofinality is non-measurable as a consequence of M's minimality. Hence, we are done if we can show that M rebuilds itself below λ.

Let N be some non-dropping iterate of M and let $\kappa < \lambda^*$ where λ^* is the image of λ under the iteration embedding. It is enough to show that if $\kappa < \beta < \lambda^*$ is $({<}\lambda^*, \alpha)$-strong where, and this seems crucial, $\alpha \leq \Delta_x^{\lambda^*}$ for some $x \in N|\beta$, then β is $({<}\lambda^*, \alpha)$-strong in $P := \mathbb{L}_\kappa^N$.

Note that if $x \in y$ then $\Delta_x^{\lambda^*} \leq \Delta_y^{\lambda^*}$, so there are actually unboundedly many cardinals $\beta < \lambda^*$ which are $({<}\lambda^*, \Delta_x^{\lambda^*})$-strong. This justifies our choice of β above.

Let $\gamma < \alpha$ and $\eta < \lambda^*$. Fix s such that $\gamma = \delta_x^{\lambda^*}(s)$. By assumption there is some extender with critical point β strong past $(\eta)^{+\omega}$ such that β is $({<}\lambda^*, \gamma)$-strong in $N^* := \mathrm{Ult}(N; E)$. Using induction one can assume that β is $({<}\lambda^*, \delta_x^{\lambda^*}(s))$-strong in $i_E(P)$. By the claim in the proof of Theorem 1.7 in [8] we have that the trivial completion of $E^* := E \cap ([\eta]^{<\omega} \times P)$ is on the sequence of P and there exists $\sigma : \mathrm{Ult}(P, E^*) \to i_E(P)$ with critical point at least η which without loss of generality is bigger than the rank of s.

Because β is $({<}\lambda^*, \gamma)$-strong in $i_E(P)$ it will be $({<}\lambda^*, \gamma)$-strong in $\mathrm{Ult}(P; E^*)$ using

$$\gamma = \delta_x^{\lambda^*}(s) = \sigma(\delta_x^{\lambda^*}(s)).$$

⊣

LEMMA 31. *Assume that $M := M^{\#}_{>\theta_{\aleph_2}}$ exists and has a good (ω_1+1)-iteration strategy. Then M together with λ its distinguished limit of Woodin cardinals and hyperstrong cardinals form a tractable pair.*

The proof is essentially the same as the previous lemma. We leave it to the reader.

§5. Derived models of mice with many hyperstrong cardinals. At this point we need to combine concepts from sections 1 and 2, but to do this those notions need some slight revisions: let M be a (ω_1+1)-iterable countable premouse and let $\mathcal{P} \in M$ be a fine-structural model, usually a HOD-premouse. Let $\alpha := \mathrm{Card}^M(\mathcal{P})$ and assume $\mathcal{P}$ has a (α^+, α^+)-iteration strategy Σ in M with hull condensation.

As a general fact about hull condensation, if Σ extends to a (β, β)-strategy in M with hull condensation, then it extends uniquely. We will generally confuse Σ with any such extensions it might posess.

We want to form a fully backgrounded hybrid-construction $\mathbb{F}_\kappa^{\Sigma,M}$ where $\kappa > (\alpha^+)^M$. The construction differs only in two minor ways: firstly we index branches given by Σ at any point in accordance with our index scheme, secondly we require that any extender has a background certificate that moves Σ to itself.

There is no scarcity of such extenders, any extender that produces an ultrapower that is closed under α-sequences in M will do. (A tree $\mathcal{T}$ is by Σ if and only if every α-sized hull is by $\Sigma \restriction M||(\alpha^+)^M$ which is not moved by the embedding, the ultrapower has all α-sized hulls of any given $\mathcal{T}$.)

Generally, we cannot expect that Σ will extend to an $(\mathrm{On}, \mathrm{On})$-strategy over M. In case we cannot fulfill our commitment at any stage to index a branch, we will terminate the construction. In order to evade the possibility of such a failure we will require that the insertion of the next branch can only be triggered at a level of the construction that is *closed under sharps*. (A fine structural model $N := (N; \in, \vec{E}^N, \dots)$ is closed under sharps, if and only if for all ordinals $\alpha \in N$ there exists $\beta < \mathrm{On} \cap N$ such that $(\vec{E}^N)_\beta \neq \emptyset$ and $\mathrm{crit}((\vec{E}^N)_\beta) > \alpha$.)

The following situation is typical: Σ extends to a (λ, λ)-iteration strategy over M where λ is a limit of M-inaccessible cardinals. We do have that no extender properly on the M-sequence (that is excluding the top extender) has index larger than λ. As a result of this we will never be asked to commit to indexing a branch we do not have as no level of the construction of size at least λ will be closed under sharps. Those levels are the only levels where we might be asked to index a branch for a tree that is not in $M||\lambda$.

Usually, Σ will extend to a (ω_1, ω_1)-strategy in V, a code of which will be Suslin-co-Suslin captured over M. In that case $\mathbb{F}_\kappa^{\Sigma,M}$ will be a Σ-hybrid mouse with all the properties we expect from its regular mouse counterpart: universality, downward absouluteness of Woodinness, and so forth.

On the topic of the locally backgrounded construction $\mathbb{L}_\kappa^{\Sigma,M}$ we note two things: a hull of any valid certificate, that is a certificate where the extenders move Σ to itself, will also be a valid certificate on account of hull condensation and the copy construction; Σ is correctly computed at any cardinal intial segment of M extending $M||(\alpha^+)^M$ and so is the validity of certificates.

With this it is not hard to see that $\mathbb{L}_\kappa^{\Sigma,M}$ will be a Σ-hybrid mouse with the required downwards absoluteness of large cardinal properties.

This sums up the material we require from section 1. In section 2 we did already mention that HOD-pairs can exist outside of determinacy models, but we did not discuss how to handle subsidiary notions, a crucial example being fullness preservation. We are going to fill in this gap now.

DEFINITION 32 (ZFC). $(\mathcal{Q}, \Lambda)$ is a HOD-pair at κ iff:

(a) $\mathcal{Q} \in H_\kappa$,
(b) $(\mathcal{Q}, \Lambda)$ is a HOD-pair,
(c) Λ is a (κ, κ)-iteration strategy with hull condensation,
(d) Λ extends into a (κ, κ)-iteration strategy with hull condensation in any $<\kappa$-generic extension.

DEFINITION 33. Let M be a countable fine-structural model and let $a \in M$ be a self-wellordered set, that is a is well-orderable in $J_1(a)$. Working in M, let λ be an inaccessible cardinal or a limit of such cardinals. Let Σ be a (λ, λ)-iteration strategy and assume that $\mathbb{F}_{\kappa}^{\Sigma, M||\lambda}(a)$ exists. Set $\mathrm{Lp}^{\lambda,\Sigma}(a) := \mathbb{F}^{\Sigma,M||\lambda}(a)||(a^{+})^{\mathbb{F}^{\Sigma,M||\lambda}(a)}$.

DEFINITION 34. Let M be a premouse, κ an inaccessible cardinal in M or a limit of such cardinals. A HOD-pair at κ in M, $(\mathcal{P}, \Sigma)$, is κ-fullness preserving if and only if for all iterations $\mathcal{T}$ by Σ with last model $\mathcal{Q}$ and iteration embedding $\iota^{\mathcal{T}} : \mathcal{P} \to \mathcal{Q}$ in H_κ we have $\mathrm{Lp}^{\kappa,\Sigma_{\mathcal{Q}(\alpha)^{-},\mathcal{T}}}(Q||\eta) \subseteq Q$ for all cardinal cutpoints η of $\mathcal{Q}$ and α minimal with $\eta \in \mathcal{Q}(\alpha)$.

DEFINITION 35. Let M be a premouse, κ an inaccessible cardinal in M or a limit of such cardinals. We say M captures $(\mathcal{Q}, \Lambda)$ below κ if and only if $(\mathcal{Q}, \Lambda)$ is a HOD-pair at κ over M and for every $g \subset \mathrm{Col}(\omega, \mathrm{Card}(Q)^{+})$ generic over M, Λ uniquely extends to a $<\kappa$-universally Baire set over $M[g]$ and the unique extension of Λ has branch condensation and is κ-fullness preserving in $M[g][h]$ where h is $<\kappa$-generic over $M[g]$.

Remark. One: if κ is $<\lambda$-strong and λ is a limit of Woodin cardinals, then any $(\mathcal{Q}, \Lambda)$ captured below κ is captured below any limits of Woodin cardinals in between κ and λ. This is a result of the universality of the background construction as a result of which all the levels of $\mathrm{Lp}^{\lambda,\Lambda}(a)$ will occur in the construction before the least Woodin cardinal larger than the rank of a.

Two: if λ is a limit of Woodin cardinals and $(\mathcal{Q}, \Lambda)$ is captured below λ, then Λ extends to a (ω_1, ω_1)-strategy that is Suslin-co-Suslin captured over M below λ (assuming M has a good $(\omega_1 + 1)$-iteration strategy).

LEMMA 36. *Let (M, λ) be a tractable pair. Let $\Gamma := \mathcal{P}(\mathbb{R}) \cap D(M, \lambda)$. Let N be a non-dropping iterate of M and let $a \in N||\lambda^*$ be self-wellordered (λ^* is the image of λ). Let $(\mathcal{P}, \Sigma)$ be a* HOD*-pair that is captured over N below λ^*. Then* $\mathrm{Lp}^{\Gamma,\Sigma}(a) = (\mathrm{Lp}^{\lambda,\Sigma})^{N}(a)$.

PROOF. For the one direction let $Q \trianglelefteq \mathrm{Lp}^{\Gamma,\Sigma}(a)$. Iterate N to N^* using extenders with critical points above $\mathrm{rank}(a)$ such that $D(M, \lambda)$ can be realized over N^*. (We have to leave V for this, but that is fine.) Q is definable from ordinal parameters and $(\mathcal{P}, \Sigma)$ (which is represented by trees in N^*), so by the homogeneity of the collapse $Q \in N^*$ and N^* is closed under its (Q's) unique iteration strategy. By elementarity the same is true in N. By universality of the background construction in the presence of Woodin cardinals we then have $Q \trianglelefteq \mathbb{F}^{N,\Sigma}(a)$ as required (see section 2.3 of [4]).

For the other direction let $Q \trianglelefteq \mathbb{F}^{N,\Sigma}(a)$. First note that Q is iterable because it inherits an iteration strategy from $\mathbb{F}^{N,\Sigma}(a)$ which in turn inherits it from N. The fact that N captures this iteration strategy follows from the proof of Lemma 5.5 in [4]. (Alternatively, consult [7] Theorem 5.1 .) $\dashv$

THEOREM 37. *Assume that there is no inner model containing all the reals such that a Wadge initial segment satisfies "$AD_{\mathbb{R}} + \Theta$ regular". Assume that $M := M^{\#}_{\theta_{\aleph_2}}$ exists and has a good $(\omega_1 + 1)$-iteration strategy, let λ be M's distinguished limits of Woodin cardinals and let $x \in V_\lambda^M$.*

Inside of M let κ be the least $(<\lambda, \Delta^\lambda_x)$-strong, $(\mathcal{P}, \Sigma^{\mathcal{P}})$ be the direct limit under coiteration of all HOD*-pairs which are captured at some $<\lambda$-strong cardinal below κ. Let $\Gamma := \mathcal{P}(\mathbb{R}) \cap D(M, \lambda)$. Let $\mathcal{P}^*$ be the $\mathrm{Lp}^{\Gamma, \Sigma^{\mathcal{P}}}$-closure of $\mathcal{P}$.*

Then $\mathcal{P}^*$ does not project below $\mathrm{On} \cap \mathcal{P}$, and has a λ-fullness preserving iteration strategy Σ with branch condensation. Σ is Suslin-co-Suslin captured over M, and $\pi^\Sigma_\infty(\lambda^{\mathcal{P}^*}) \geq \Delta^\lambda_x$ where $\pi^\Sigma_\infty : \mathcal{P}^* \to \mathrm{HOD}^{D(M,\lambda)}$ is the HOD*-limit embedding by Σ.*

Proof. Note that κ is a regular limit of $<\lambda$-strong cardinals and Woodin cardinals. Hence $D(M, \kappa) \models \mathrm{AD}_{\mathbb{R}} + \mathrm{DC}$ (Theorems 2.10 and 2.11 from [7], see also [9]). From this using the HOD-analysis (,see section 2,) we can deduce that $\lambda^{\mathcal{P}}$ has to be a limit ordinal and its $\mathcal{P}^*$-cofinality is the critical point of a total on $\mathcal{P}$ extender.

We note here that any pair $(\mathcal{Q}_0, \Lambda_0), (\mathcal{Q}_1, \Lambda_1)$ both captured at κ over M can be compared successfully in M. That is because κ is a limit of Woodin cardinals, and therefore a background construction will reach $M_1^{\Lambda_0, \Lambda_1, \#}$, a HOD-construction of which will reach a common iterate.

We can and do assume that $\lambda^{\mathcal{P}^*} \in \mathcal{P}(0)$ and $\pi_\infty^{\Sigma^{\mathcal{P}}_{\mathcal{P}(0)}}(\lambda^{\mathcal{P}^*}) \leq \Delta^\lambda_x$. If either of these fail, then we can show with what was proven above that the derived model contains some HOD-pair (Q, Σ) such that $\pi^\Sigma_\infty(\lambda^Q) \geq \Delta^\lambda_x$ which happens to be what we are after in the first place.

First we will show that any iteration tree $\mathcal{T}$ based on $\mathcal{P}$, that is according to $\Sigma^{\mathcal{P}} = \bigoplus_{\alpha < \lambda^{\mathcal{P}}} \Sigma^{\mathcal{P}}_{\mathcal{P}(\alpha)}$, considered as an iteration tree on $\mathcal{P}^*$ has a wellfounded last model, call it $\mathcal{Q}^*$, and $\mathcal{Q}^*(\alpha)$ has a Γ-fullness preserving iteration strategy with branch condensation for all $\alpha < \lambda^{\mathcal{Q}^*}$.

So, let us fix one such iteration tree $\mathcal{T}$ on $\mathcal{P}$. Without loss of generality $\mathcal{T}$ is based on the window $\left[\delta^{\mathcal{T}}_\alpha, \delta^{\mathcal{T}}_{\alpha+1}\right]$ (any tree is ultimately just a stack of such trees). Let $j : M \to N$ be an elementary embedding with critical point κ such that $\mathcal{T} \in N$ and $j(\kappa) > \mathrm{rank}(\mathcal{T})$. Because $\Sigma^{\mathcal{P}}_{\mathcal{P}(\alpha+1)}$ is Suslin-co-Suslin captured over M we have

$$j(\Sigma^{\mathcal{P}}_{\mathcal{P}(\alpha+1)}) = \Sigma^{\mathcal{P}}_{\mathcal{P}(\alpha+1)} \cap N$$

and thus there exist iteration embeddings

$$\mathcal{P}(\alpha+1) \to^\pi \mathcal{Q}^*(\alpha+1) \to^\sigma j(\mathcal{P}(\alpha+1))$$

in N. We define $\sigma^* : \mathcal{Q}^* \to j(\mathcal{P}^*)$ by

$$\pi(f)(a) \to j(f)(\sigma(a))$$

where $f \in \mathcal{P}^*$ and $a \in \mathcal{Q}^*(\alpha+1)$.

Claim 1. σ^* is elementary.

PROOF OF CLAIM.

$$\begin{aligned}&\mathcal{Q}^* \models \varphi(\pi(f)(a))\\ \Leftrightarrow & j(\mathcal{Q}^*) \models \varphi(j(\pi)(j(f))(j(a)))\\ \Leftrightarrow & j(\mathcal{Q}^*) \models \varphi(j(\pi)(j(f))(\pi_{j(\mathcal{P}(0)),j(\mathcal{Q}(0))}(\sigma(a)))\\ \Leftrightarrow & j(\mathcal{Q}^*) \models \varphi(j(\pi)(j(f))(j(\pi_{\mathcal{P}(0),\mathcal{Q}(0)})(\sigma(a)))\\ \Leftrightarrow & j(\mathcal{P}^*) \models \varphi(j(f)(\sigma(a)))\end{aligned}$$

□

So, not only is $\mathcal{Q}^*$ well-founded but each of its proper HOD-initial segments is iterable. Because $\lambda^{\mathcal{Q}^*}$ is a limit ordinal we even know that the strategies of those segments have branch condensation (see [4] Lemma 3.29), because they inherit that from $j(\mathcal{P})$.

It remains to be seen that this strategy, or more accurately, this join of strategies which we will call $\Sigma^{\mathcal{Q}}$ is fullness preserving. For that we first need to show that no level of $\mathcal{P}^*$ projects below $\mathrm{On} \cap \mathcal{P}$.

Claim 2. No level of $\mathcal{P}^*$ projects below $\mathrm{On} \cap \mathcal{P}$.

PROOF OF CLAIM. Assume it does. Let $\mathcal{R}$ the least initial segment of $\mathcal{P}^*$ witnessing it. Let $\alpha < \lambda^{\mathcal{P}}$ be (minimal) such that $\rho_\omega(\mathcal{R}) < \delta_\alpha^{\mathcal{P}}$. If $\lambda^{\mathcal{P}}$ is singular in $\mathcal{R}$ we also want to assume that $\delta_\alpha^{\mathcal{P}} > \mathrm{cof}^{\mathcal{R}}(\lambda^{\mathcal{P}})$. We can do so by increasing α if need be.

Consider $\mathcal{Q}$, the transitive collapse of $\mathrm{Hull}_{m+1}^{\mathcal{R}}(\mathcal{P}(\alpha) \cup \{p_{m+1}^{\mathcal{R}}\})$, where m is minimal with $\rho_{m+1}(\mathcal{R}) < \delta_\alpha^{\mathcal{P}}$. We want to show that $\mathcal{Q}$ has an iteration strategy Λ in Γ, considered as a $\Sigma_{\mathcal{P}(\alpha)}^{\mathcal{P}}$-HOD mouse. For simplicity's sake we will only consider normal trees which are, in fact, sufficient for our needs.

We will define Λ as the strategy that picks the unique branches that have Q-structures given by the appropriate fully backgrounded construction, (so if $\mathcal{T}$ is a tree on the window $\left(\delta_\beta^{\mathcal{Q}}, \delta_{\beta+1}^{\mathcal{Q}}\right)$, then Q-structures are given by $\mathbb{F}^{\Sigma_{\mathcal{Q}(\beta)}^{\mathcal{Q}}, M}(\cdot)$,) or are j-realizable where $j : M \to N$ is a sufficiently strong embedding on M with critical point κ.

Our previous considerations can be applied here to show that the pullback of $\Sigma^{\mathcal{P}}$ onto $\mathcal{Q}$ gives the existence of such branches, but we do need them to be unique. Fix some $\mathcal{T}$ according to Λ and b, c both cofinal branches through $\mathcal{T}$ that fit the requirement to be $\Lambda(\mathcal{T})$.

It is easy to see that neither b nor c can have Q-structures, hence there must be realization embeddings $\sigma_b : \mathcal{M}_b^{\mathcal{T}} \to j(\mathcal{Q})$ and $\sigma_c : \mathcal{M}_c^{\mathcal{T}} \to j(\mathcal{Q})$ such that $j = \sigma_b \circ \iota_b^{\mathcal{T}} = \sigma_c \circ \iota_b^{\mathcal{T}}$ for some fixed $j : M \to N$.

Let $x_b := \mathrm{Hull}_{m+1}^{\mathcal{M}_b^{\mathcal{T}}}(\mathcal{P}(\alpha) \cup \{p_{m+1}^{\mathcal{M}_b^{\mathcal{T}}}\}) \cap \delta(\mathcal{T}) \subset \mathrm{ran}(\iota_b^{\mathcal{T}})$ and $x_c := \mathrm{Hull}_{m+1}^{\mathcal{M}_c^{\mathcal{T}}}(\mathcal{P}(\alpha) \cup \{p_{m+1}^{\mathcal{M}_c^{\mathcal{T}}}\}) \cap \delta(\mathcal{T}) \subset \mathrm{ran}(\iota_c^{\mathcal{T}})$. x_b and x_c are both cofinal in $\delta(\mathcal{T})$; this is because we are never introducing any new Woodin cardinals in our iteration as the cofinality of $\lambda^{\mathcal{P}}$ is never moved.

Now both $\mathcal{M}_b^{\mathcal{T}}$ and $\mathcal{M}_c^{\mathcal{T}}$ inherit an iteration strategy via $j(\Sigma^{\mathcal{P}})$, hence we can compare them into a common model $\mathcal{Q}^*$ via a co-iteration that is entirely above

$\delta(\mathcal{T})$. We then have

$$x_b = \mathrm{Hull}_{m+1}^{\mathcal{Q}^*}(\mathcal{P}(\alpha) \cup \{p_{m+1}^{\mathcal{Q}^*}\}) \cap \delta(\mathcal{T}) = x_c.$$

By Lemma 22 we conclude that $b = c$, as wanted.

Having defined Λ, it is not hard to show that $\Lambda \in D(M, \lambda)$. An in-depth look at a similar argument can be found towards the end of this proof, so we will omit the details here.

Now let a be the least set missing from $\mathcal{R}$ which is the least set missing from $\mathcal{Q}$. a is definable in $D(M, \lambda)$: for a Turing-cone of reals x, a is the least set missing from the least anomalous HOD-premouse that appears in the $\Gamma(\mathcal{Q}, \Lambda)$-HOD-mouse construction relative to $\Sigma^{\mathcal{P}}_{\mathcal{P}(\alpha)}$ in any coarse mouse (O, Φ) capturing $\Gamma(\mathcal{Q}, \Lambda)$ and containing x.

This definition is $\mathrm{OD}^{D(M,\lambda)}_{\Sigma^{\mathcal{P}}_{\mathcal{P}(\alpha)}}$, because $\Gamma(\mathcal{Q}, \Lambda)$ is definable from its Wadge degree. By mouse capturing we then have $a \in \mathrm{Lp}^{\Gamma, \Sigma^{\mathcal{P}}_{\mathcal{P}(\alpha)}}(\mathcal{P}(\alpha)) \subset \mathcal{P}$. Contradiction! □

Now we can form $\mathcal{P}^{**} := \mathbb{L}^{\Sigma^{\mathcal{P}}, M}(\mathcal{P})$; from the claim we can deduce that no level of the construction projects below $\mathrm{On} \cap \mathcal{P}$. The proof above that any $\mathcal{P}(0)$-based iteration lifts applies to this model, too. In fact, the iteration stretches onto $L(\mathcal{P}^{**})$. Let $\mathcal{Q}^{**}$ be the model that results from $\mathcal{T}$ being applied to $\mathcal{P}^{**}$.

So, there is some $j : M \to N$ such that $L(\mathcal{Q}^{**})$ embeds into $L(j(\mathcal{P}^{**}))$. Also, $L(\mathcal{Q}^{**})$ satisfies the large cardinal requirement of being $M^{\#}_{\theta_{\aleph_2}}$ (after appending the top measure of M). Thus M can be rebuilt inside of $L(\mathcal{P}^{**})$, as well as $L(\mathcal{Q}^{**})$ and $L(j(\mathcal{P}^{**}))$. If we look at the proof of Theorem 3.6 in [7] we get:

$$D(M, \lambda) = D(L(\mathcal{P}^{**}), \lambda) = D(L(\mathcal{Q}^{**}), \lambda) = D(L(j(\mathcal{P}^{**})), \lambda).$$

By the first equality we have that $L(\mathcal{P}^{**})$ believes "$\Sigma^{\mathcal{P}}_{\mathcal{P}(\alpha)}$ is a fullness preserving iteration strategy with branch condesation in my derived model for all $\alpha < \lambda^{\mathcal{P}^*}$". Let $\alpha < \lambda^{\mathcal{Q}^*}$, we then have by elementarity that $\Sigma^{\mathcal{Q}}_{\mathcal{Q}^*(\alpha)}$ is fullness preserving and has branch condensation. This finishes the first part of the proof.

We have just shown that all the iteration strategies appearing in iterations of $\mathcal{P}^*$ are in the derived model and fullness preserving. This immediately suggests a way to define a strategy for $\mathcal{P}^*$: iterate according to the stretched strategies produced by applying extenders with critical point κ. Unfortunately, there is no guarantee that $\Sigma^{\mathcal{Q}}$ is definable over M or that it gives a coherent strategy.

Note that so far we have only used that κ is $<\lambda$-strong. It is now opportune to cash in on our large cardinal assumption. We will show that the hyperstrong cardinals can be used to stitch together the $\Sigma^{\mathcal{Q}}$ into a full iteration strategy. Every degree of strength corresponds to catching one extra iteration strategy.

The basic idea is standard: consider a tree $\mathcal{T}$, it is easy to see that we can think of $\mathcal{T}$ as a combination of two trees $\mathcal{T}^-$ which is on $\mathcal{P}$ exclusively and $\mathcal{T}^+$ which is entirely above $\mathcal{P}$. We stretch $\mathcal{P}^*$ using some strong embedding $j : M \to N$ such that $\mathcal{T} \in N$. Our prior considerations yield an embedding $\iota : \mathcal{P}^* \to \mathcal{Q}^*$ which can be understood as the iteration embedding on $\mathcal{P}^*$ by $\mathcal{T}^-$, and a realization embedding $\sigma^* : \mathcal{Q}^* \to j(\mathcal{P}^*)$. We very much desire that $\sigma^* \restriction \iota(\mathcal{P})$ be the iteration embedding of $\iota(\mathcal{P})$ into some intial segment of $j(\mathcal{P})$

by $\Sigma^{\mathcal{Q}}$. Given that we could then identify the branch through $\mathcal{T}^+$ as the unique σ^*-realizable branch. By absoluteness it is enough to prove that this embedding exists somewhere in the universe, say the derived model. By the above every level of $\mathcal{Q}^*$ does have an iteration strategy in the derived model and thus we can try to line up $\iota(\mathcal{P})$ and $j(\mathcal{P})$. A priori, there is no guarantee that $j(\mathcal{P})$ does not move in that co-iteration.

Let us consider the first steps of such a co-iteration procedure. We do know that N is closed under $\Sigma^{\mathcal{P}}$ as $\mathcal{P}$ is still the local HOD limit at κ in N. Thus working in N, given some $\bigoplus_{\alpha<\lambda^{\mathcal{P}}} j(\Sigma^{\mathcal{P}}_{\mathcal{P}(\alpha)})$ suitable mouse $\mathcal{R}$ of size $<j(\kappa)$, we can pseudo-co-iterate $\mathcal{R}$ and $\mathcal{Q}^*(\lambda'+1)$ into a common model where $\lambda' := \sup(\iota\text{"}\left[\lambda^{\mathcal{P}}\right])$. Thus there exists in the derived model an embedding from $\mathcal{Q}^*(\lambda'+1)$ into $j(\mathcal{P})(\lambda''+1)$ where $\lambda'' := \sup(j\text{"}\left[\lambda^{\mathcal{P}}\right])$.

The fact that we can carry along $L(\mathcal{P}^{**})$ during this process will guarantee that the resulting branch is unique. By an absoluteness argument we will get that M is closed under $\Sigma^{\mathcal{Q}}_{\mathcal{Q}^*(\lambda'+1)}$. We cannot prove that N is closed under $\Sigma^{\mathcal{Q}}_{\mathcal{Q}^*(\lambda'+1)}$ as $j\restriction\mathcal{P}^*$ might not be in N and we can therefore not define in N the tree searching for realizable embeddings. This means that the above argument cannot be repeated at $\lambda'+2$ because N might not have the requisite Q-structures.

If, on the other hand, κ were still strong in N we could relativize the above argument and get N closed under $\Sigma^{\mathcal{Q}}_{\mathcal{Q}^*(\lambda'+1)}$ and thus get M closed under $\Sigma^{\mathcal{Q}}_{\mathcal{Q}^*(\lambda'+2)}$. One might thus expect that if κ is $(<\lambda,\alpha)$-strong in M we might get M closed under $\Sigma^{\mathcal{Q}}_{\mathcal{Q}^*(\lambda'+\alpha)}$.

We will now define Σ a strategy for $\mathcal{P}^*$ extending $\Sigma^{\mathcal{P}}$. Assume that Σ has already been defined for trees below some fixed countable length. Let $\vec{\mathcal{S}}$ be some stack of normal trees of less than that fixed length according to Σ. We shall assume that $\vec{\mathcal{S}}$ can be written as $\vec{\mathcal{U}}^\frown\vec{\mathcal{T}}$ where $\vec{\mathcal{U}}$ is based on $\mathcal{P}$ and $\vec{\mathcal{T}}$ is a tree on $\mathcal{Q}^*$, the last model of $\vec{\mathcal{U}}$ if it exists, above $\mathcal{P}$. We mean that $\vec{\mathcal{T}}$ lives on the new Woodin cardinals we created when we stretch the iteration embeddings of $\vec{\mathcal{U}}$ onto $\mathcal{P}^*$.

We require that $\vec{\mathcal{U}}$ be by $\Sigma^{\mathcal{P}}$. If $\mathcal{Q}^*$ exists let $\langle \mathcal{T}_\xi : \xi < \eta'\rangle$ be the essential components of $\vec{\mathcal{T}}$, in the following we shall identify $\vec{\mathcal{T}}$ with its essential components. We assume that it has a last component, say $\eta' = \eta+1$. Let $\mathcal{R}^* := \mathcal{M}^{\vec{\mathcal{T}}}_\eta$, there is some $\xi < \lambda^{\mathcal{R}^*}$ such that $\mathcal{T}_\eta$ is a normal tree on $\mathcal{R}^*(\xi+1)$. Let $\delta := \pi_{\mathcal{R}^*(0),\infty}(\xi)$ where $\pi_{\mathcal{R}^*(0),\infty}$ is the direct limit embedding given by the appropriate $\Sigma^{\mathcal{P}}$-tail. Let $j : M \to N$ be an embedding with critical point κ such that $\vec{\mathcal{S}} \in N||j(\kappa)$ and κ is $(<\lambda,\delta)$-strong in N. Let $\sigma^* : \mathcal{Q}^* \to j(\mathcal{P}^*)$ be the $\iota^{\vec{\mathcal{U}}}$-realization embedding. We assume that there is some $\tau : \mathcal{R}^* \to j(\mathcal{P}^*)$ such that $\tau \circ \iota^{\vec{\mathcal{T}}}_{0,\eta} = \sigma^*$, and that N is definably closed under $\Sigma^{\mathcal{Q}}_{\mathcal{R}^*(\xi),\vec{\mathcal{T}}\restriction\eta}$ and $\tau\restriction\mathcal{R}^*(\xi)$ is the iteration embedding, otherwise Σ is undefined on $\vec{\mathcal{S}}$.

Set $b = \Sigma(\vec{\mathcal{S}})$ iff:

(1) if $\vec{\mathcal{T}} = \emptyset$ then $b = \Sigma^{\mathcal{P}}(\vec{\mathcal{U}})$; otherwise continue;
(2) if $\mathcal{T}_\eta$ has a fatal drop then let b be according to the unique iteration strategy; if $\rho_\omega(\mathrm{Lp}^{\Gamma,\Sigma^{\mathcal{Q}}_{\mathcal{R}^*(\xi),\vec{\mathcal{T}}\restriction\eta}}(\mathcal{M}(\mathcal{T}_\eta))) < \delta(\mathcal{T}_\eta)$ or reaches a level Q over which $\delta(\mathcal{T}_\eta)$

is not definably Woodin, then b is the unique branch such that $Q = Q(b, \mathcal{T}_\eta)$, otherwise continue;

(3) b is the unique branch such that there exists $\pi : \mathcal{M}_b^{\mathcal{T}_\eta}(\xi+1) \to j(P^*)(\tau(\xi+1))$ such that $\pi \circ (\iota_b^{\mathcal{T}_\eta} \restriction \mathcal{R}^*(\xi+1)) = \tau \restriction \mathcal{R}^*(\xi+1)$.

Claim 3. Σ is both well defined and total on its domain.

PROOF OF CLAIM. First, a quick comment: there are two conditions in the proof we have to inductively verify. The first condition can be verified by noticing that in "(3)" because both τ and $\iota_b^{\mathcal{T}_\eta}$ were total on $\mathcal{R}^*$, we can always stretch π to be a total embedding on $\mathcal{M}_b^{\mathcal{T}_\eta}$. If in "(2)", Player I (in the iteration game) is able to open a new round of normal iteration afterwards, by necessity there was no dropping on b. The proof will then show that an embedding as in "(3)" exists. As concerns the second condition, the following proof will show that a suitably restricted version of the above definition will give a definition of $\Sigma^{\mathcal{Q}}_{\mathcal{R}^*(\xi),\vec{\mathcal{T}}\restriction\eta}$ inside N.

Let now $\vec{\mathcal{S}}$ be as above. We will show that the required branch exists in M. With what we have proved above it is easy to see that clause "(1)" is unproblematic. If we are in clause "(2)" then we refer to Lemma 36 applied in N to see that background constructions will reach the required Q-structures keeping in mind that $\mathcal{R}^*(\xi+1)$ does in fact have an iteration strategy in the derived model. By Lemma 22 the branches given by these Q-structure are unique and thus by an absoluteness argument these branches are elements of M. If we are in clause "(3)" on the other hand we will have to show two things: the existence of a branch together with a realizing embedding and the uniqueness of such a branch.

We start with uniqueness. Let us fix a b and some π as above. We will want to show that $\mathrm{ran}(\iota_b^{\mathcal{T}_\eta})$ contains a specific set cofinal in $\delta_{\xi+1}^{\mathcal{M}_b^{\mathcal{T}_\eta}}$ which in turn fixes b.

First note that using σ^* as an embedding on the whole of $L(\mathcal{Q}^{**})$ means that τ,$\iota_b^{\mathcal{T}_\eta}$ and π extend up to class size models too. We can fix an infinite set of indiscernibles I all of which are fixed by all relevant embeddings, say an infinite collection of uncountable cardinals. Let $L(B)$ be the extension of $\mathcal{M}_b^{\mathcal{T}_\eta}$. We want to show that points definable from I and $\mathrm{ran}(\iota_b^{\mathcal{T}_\eta} \restriction \mathcal{R}^*(\xi))$ are cofinal in $\delta_{\xi+1}^{\mathcal{M}_b^{\mathcal{T}}}$.

As iteration embeddings are continous at Woodin cardinals it is enough to show that points definable from I and $\mathcal{R}^*(\xi)$ over $L_{\sup(I)}(\mathcal{R}^{**})$ are cofinal in $\delta_{\xi+1}^{\mathcal{R}^*}$!

Assume not! Let $\zeta < \delta_{\xi+1}^{\mathcal{R}^*}$ be the supremum of points definable from the above parameters. Let $L_\alpha(\mathcal{R}')$ be the tranistive collapse of $\mathrm{Hull}^{L_{\sup(I)}(\mathcal{R}^{**})}(\zeta \cup I)$. By a standard result of Baumgartner's $\mathrm{Hull}^{L_{\sup(I)}(\mathcal{R}^{**})}(\zeta \cup I) \cap \delta_{\xi+1}^{\mathcal{R}^*} = \zeta$, so ζ is in $L_\alpha(\mathcal{R}')$ the least Woodin above $\delta_\xi^{\mathcal{R}^*}$. Note also that $\mathcal{R}'$ is still a $\Sigma^{\mathcal{Q}}_{\mathcal{R}^*(\xi),\vec{\mathcal{T}}\restriction\eta}$-hybrid mouse by hull condensation for strategies.

Now we have that $L_{\sup(I)}(\mathcal{R}^{**})$ believes that "$\delta_{\xi+1}^{\mathcal{R}^*}$ is Woodin in $\mathrm{Lp}^{\Sigma^{\mathcal{Q}}_{\mathcal{R}^*(\xi),\vec{\mathcal{T}}\restriction\eta}}(\mathcal{R}^*||\delta_{\xi+1}^{\mathcal{R}^*})$ as computed in my derived model" but also "ζ is not Woodin in $\mathrm{Lp}^{\Sigma^{\mathcal{Q}}_{\mathcal{R}^*(\xi),\vec{\mathcal{T}}\restriction\eta}}(\mathcal{R}^*||\zeta)$

as computed in my derived model". By elementarity then $L_\alpha(\mathcal{R}')$ believes "ζ is Woodin in $\mathrm{Lp}^{\Sigma^{\mathcal{Q}}_{\mathcal{R}^*(\xi),\vec{\mathcal{T}}\restriction\eta}}(\mathcal{R}'||\zeta)$ as computed in my derived model".

But we have that $L(\mathcal{R}^{**})||\zeta = L_\alpha(\mathcal{R}')||\zeta$ and they both realize the same derived model because they both rebuild $M^{\#}_{\theta_{\aleph_2}}$. Contradiction!

We now want to show the existence of such an embedding. The proof proceeds in two steps. First we show that there exists an iteration embedding from both $\mathcal{R}^*(\xi+1)$ and the presumed final $M_b^{\mathcal{T}_\eta}(\xi+1)$, which we can identify in N, into $j(\mathcal{P}^*)(\tau(\xi+1))$ and secondly that it is identical with τ.

A crucial fact here is that $\mathcal{R}^*(\xi{+}1)$ can be pseudo-iterated inside of $j(L(\mathcal{P}^{**}))$, using that it is closed under $\Sigma^{\mathcal{Q}}_{\mathcal{R}^*(\xi),\vec{\mathcal{T}}\restriction\eta}$ and thus closed under $\Sigma^{\mathcal{Q}}_{\mathcal{R}^*(\xi),\vec{\mathcal{T}}\restriction\eta} - Q$-structures by a relativized version of Lemma 36. So $\mathcal{R}'(\xi{+}1) := \mathcal{M}_b^{\mathcal{T}_\eta}(\xi{+}1) \in N$. One can show that working inside of the derived model $\mathcal{R}'(\xi+1)$ iterates into $j(\mathcal{P}^*)(\tau(\xi+1))$. This is because in N, $\mathcal{R}'(\xi)$ iterates into $j(P^*)$ up to level $\tau(\xi)$ and thus $\mathcal{R}'(\xi+1)$ which has a fullness preserving strategy in the derived model will co-iterate with $j(\mathcal{P}^*)(\tau(\xi+1))$ to a common model which we claim to be $j(\mathcal{P}^*)(\tau(\xi+1))$.

N correctly identifies $\mathcal{R}'(\xi{+}1)$ as a $\Sigma^{\mathcal{Q}}_{\mathcal{R}^*(\xi),\vec{\mathcal{T}}\restriction\eta}$-suitable premouse. It can therefore track the aforementioned co-iteration using Q-structures. But then $j(\mathcal{P}^*)$ does not move as wanted being the pseudo-limit of all $\Sigma^{j(\mathcal{P})}_{\tau(\xi)}$-suitable premice in N. Of course the existence of an iteration embedding from $\mathcal{R}'(\xi+1)$ into $j(\mathcal{P}^*)(\tau(\xi+1))$ also gives an iteration from $\mathcal{R}^*(\xi+1)$ into it.

So we have our iteration embedding, but at this point we do not know if it acts on all of $L(\mathcal{R}^{**})$ or commutes with j. For that purpose we shall show that it is identical with τ.

First we will show that $\mathcal{P}^*$ is contained in $\mathrm{Hull}^{L_{\sup(I)}(\mathcal{P}^{**})}(I \cup \mathcal{P})$. For that it is enough to show that this hull is cofinal in $\mathcal{P}^*$ as then for every element in the hull we also have a surjection from $\mathcal{P}$ onto that element. Cofinality follows easily using the same argument we used to prove uniqueness of branches.

Secondly, we see that τ agrees with the iteration embedding on $\mathcal{P}$, because in fact $j \restriction P$ is an iteration embedding. It remains to be seen that the iteration embedding acts on $L(\mathcal{R}^{**})$. We have that $\Sigma^{\mathcal{P}}_{\mathcal{P}(\alpha)}$ acts on $L(\mathcal{P}^{**})$ for all $\alpha < \lambda^{\mathcal{P}}$. Realizing into $j(L(\mathcal{P}^{**}))$ is exactly how we prove this. But then by elementarity the same holds in $L(\mathcal{R}^{**})$.

So, we have now that τ is an iteration embedding. Using the fact that HOD-pair iteration strategies are commuting we thus get the commuting diagram

$$\begin{array}{ccc} \mathcal{M}_b^{\mathcal{T}_\eta}(\xi+1) & \xrightarrow{\quad\pi\quad} & j(\mathcal{P}^*)(\tau(\xi+1)) \\ & {\scriptstyle \iota_b^{\mathcal{T}_\eta}} \nwarrow \quad \nearrow {\scriptstyle \tau} & \\ & \mathcal{R}^*(\xi+1) & \end{array} .$$

So we know that a branch filling a diagram like this exists, but then by absoluteness together with its uniqueness it exists in M. □

So, we have defined a (λ, λ)-strategy Σ over M. We want to show that some natural extension of Σ gets into the derived model. Let $g \subset \mathrm{Col}(\omega, \mathcal{P}^*)$ be generic over M. Let h be $<\lambda$-generic over $M[g]$. Using essentially the same proof we can define a (λ, λ)-strategy Σ^h over $M[g][h]$. We just have to make sure that for a given tree in $M[g][h]$ our extenders are strong enough such that the ultrapower N contains a name for the tree and $g \times h$ is generic over N. This is easy to arrange.

Let η be the least M-inaccessible cardinal above λ. For any transitive $\bar{M}$ that is elementarily embeddable into $M||\eta$ with all relevant objects in the range of the embedding and some $\bar{h}$ which is $<\bar{\lambda}$-generic over $\bar{M}[g]$ where $\bar{\lambda}$ is the preimage of λ, we can define a $(\bar{\lambda}, \bar{\lambda})$-iteration strategy using the definition above. We will call that strategy $\Sigma^{\bar{M},\bar{h}}$.

We can then define trees T, U searching for $\sigma : \bar{M} \to M||\eta$ elementary with all relevant objects in the range of σ and fixing $\mathcal{P}^*$, some $\bar{h}$ that is $<\sigma^{-1}(\lambda)$-generic over $\bar{M}[g]$ and, finally, some pair of reals p such that $p \in \Sigma^{\bar{M},\bar{h}}$ (for T) or $p \notin \Sigma^{\bar{M},\bar{h}}$ (for U).

Claim 4. Let h be $<\lambda$-generic over $M[g]$. Then $(p[T])^{M[g][h]} = \Sigma^h \cap (\mathbb{R}^2) \cap M[g][h]$ and $(p[U])^{M[g][h]} = \mathbb{R}^2 \cap M[g][h] \setminus \Sigma^h$.

Proof of Claim. To see that $\Sigma^h \cap \mathbb{R} \cap M[g][h] \subseteq (p[T])^{M[g][h]}$ we just have to take the appropriate substructures of $M||\eta[g][h]$, similarly for U. It remains to be seen that T and U have empty co-projection. This in turn easily follows from the fact that whenever $\bar{M}, \bar{h}$ are as in the definition of T, U, then $\Sigma^{\bar{M},\bar{h}} = \Sigma^h \cap \bar{M}[g]\left[\bar{h}\right]$.

There are two main cases: in the first case $\Sigma^{\bar{M},\bar{h}}$ picks a branch with a Q-structure, this Q-structure is internally certified by a background construction of a generic extension of an ultrapower of $\bar{M}$'s and is thus iterable; we conclude that $\Sigma^{\bar{M},\bar{h}}$ picks the correct branch; in the second case $\Sigma^{\bar{M},\bar{h}}$ picks a branch that is realizable into an image of $\mathcal{P}^*$ by an ultrapower of $\bar{M}$, but a copy construction yields that the chosen branch is realizable into the image of $\mathcal{P}^*$ by an ultrapower of M and therefore by Σ^h. □

Let us now show that Σ and its extensions, as defined above, are indeed Γ-fullness preserving.

Claim 5. Σ and its extensions are Γ-fullness preserving.

Proof of Claim. An essential part of the construction was to show that all the strategies of HOD-inital segments appearing in iterations by Σ are fullness preserving. Because $\mathcal{P}^*$ is of limit type, the only remaing point of failure is at the top, that is in between $\mathcal{P}$ and $\mathcal{P}^*$.

Let $\mathcal{T}$ be a tree on $\mathcal{P}^*$ by Σ with last model $\mathcal{Q}^*$ and no drop on its main branch. Assume for a contradiction that some $\mathcal{M} \trianglelefteq \mathrm{Lp}^{\Gamma,\Sigma^{\mathcal{Q}}}(\iota^{\mathcal{T}}(\mathcal{P}))$ is not in $\mathcal{Q}^*$. By construction of Σ we can find some $j : M \to N$ with critical point κ and some $\sigma : \mathcal{Q}^* \to j(\mathcal{P}^*)$ such that $j \upharpoonright \mathcal{P}^* = \sigma \circ \iota^{\mathcal{T}}$. This diagram can then be lifted to $\iota^* : L(\mathcal{P}^{**}) \to L(\mathcal{Q}^{**})$, $\sigma^* : L(\mathcal{Q}^{**}) \to L(j(\mathcal{P}^{**}))$ such that $j \restriction L(\mathcal{P}^{**}) = \sigma^* \circ \iota^*$.

By construction $L(\mathcal{P}^{**})$ believes that $\mathcal{P}^*$ is full relative to $\Sigma^{\mathcal{P}}$-hybrid mice that exist in its derived model. By elementarity then $L(\mathcal{Q}^{**})$ believes the same for $\iota^{\mathcal{T}}(\mathcal{P})$ and $\Sigma^{\mathcal{Q}}$-hybrid mice. As discussed previously the derived model of $L(\mathcal{Q}^{**})$ must be $D(M,\lambda)$. Therefore $\mathcal{M} \in \mathcal{Q}^*$. Contradiction! □

So, it remains to be seen that $\pi^{\Sigma}_{\infty}(\lambda^{\mathcal{P}^*}) \geq \Delta^{\lambda}_x$. First notice that for any $j : M \to N$ with critical point κ

$$\lambda^{j(\mathcal{P}^*)} = j(\lambda^{\mathcal{P}^*}) = \pi^{\Sigma}_{\mathcal{P}(0),j(\mathcal{P}(0))}(\lambda^{\mathcal{P}^*}).$$

Let us assume for a contradiction that $\pi^{\Sigma}_{\infty}(\lambda^{\mathcal{P}^*}) = \delta < \Delta^{\lambda}_x$. Let $j : M \to N$ be a nontrivial elementary embedding such that κ is $(<\lambda, \delta)$-strong in N. Notice that $j(\pi^{\Sigma}_{\infty}(\lambda^{\mathcal{P}^*})) = \pi^{\Sigma}_{\infty}(\lambda^{\mathcal{P}^*})$ so that we can run the above proof to get $\Sigma \in D(N, j(\kappa))$ but on the other hand the direct limit of $\mathcal{P}^*$-iterates at $j(\kappa)$ will end up with $j(\lambda^{\mathcal{P}})$ many Woodin cardinals out-iterating $j(\mathcal{P})$. Contradiction! ⊣

Corollary 38. $D(M^{\#}_{\theta_{\aleph_2}}, \lambda) \models \Theta \geq \theta_{\aleph_2}$.

Proof. By the above theorem $D(M^{\#}_{\theta_{\aleph_2}}, \lambda) \models \Theta \geq \theta_{\Delta}$ where $\Delta := \sup_{x \in M||\lambda} \Delta_x$. Notice that Δ_x is exactly the least x-indiscernible greater than λ. λ, of course, is the $\aleph_1$ of the derived model. This does, in fact, make Δ the second uniform indiscernible. It is well known that the second uniform indiscernible equals $\aleph_2$ under determinacy. QED! ⊣

Question 1. *If our guess is correct and $D(M,\lambda)$ satisfies $\Theta = \theta_{\aleph_2}$ it should satisfy* DC*. Does it?*

Lemma 39. *Assume that there is no inner model containing all the reals such that a Wadge initial segment satisfies "$AD_{\mathbb{R}} + \Theta$ regular". Assume that $M := M^{\#}_{>\theta_{\aleph_2}}$ exists and has a good (ω_1+1)-iteration strategy, let λ be M's distinguished limit of Woodin cardinals .*

Inside of M let κ be the least cardinal which is $(<\lambda, \Delta^{\lambda}_x)$-strong for every $x \in (V_\kappa)^M$, $(\mathcal{P}, \Sigma^{\mathcal{P}})$ be the direct limit under coiteration of all HOD*-pairs which are captured at some $<\lambda$-strong cardinal below κ. Let $\Gamma := \mathcal{P}(\mathbb{R}) \cap D(M,\lambda)$. Let $\mathcal{P}^*$ be the* $\mathrm{Lp}^{\Gamma,\Sigma^{\mathcal{P}}}$*-closure of $\mathcal{P}$.*

Then $\mathcal{P}^*$ does not project below $\mathrm{On} \cap \mathcal{P}$*, and has a λ-fullness preserving iteration strategy Σ with branch condensation. Σ is Suslin-co-Suslin captured over M and $\pi^{\Sigma}_{\infty}(\lambda^{\mathcal{P}^*}) \geq \Delta^{\lambda}_x$ for all $x \in (V_\lambda)^M$ where $\pi^{\Sigma}_{\infty} : \mathcal{P}^* \to \mathrm{HOD}^{D(M,\lambda)}$ is the* HOD*-limit embedding by Σ.*

Proof. Let $x \in (V_\lambda)^M$. Let $j : M \to N$ be an embedding with critical point κ and $x \in (V_{j(\kappa)})^N$. We have

$$j(\pi^{\Sigma}_{\infty}(\lambda^{\mathcal{P}^*})) = \pi^{j(\Sigma)}_{\infty}(\lambda^{j(\mathcal{P}^*)}) = \pi^{\Sigma}_{\infty}(\lambda^{\mathcal{P}^*}).$$

"$\geq$" is immediate, to see "$\leq$" note that we can assume $\lambda^{\mathcal{P}^*} \in \mathcal{P}(0)$ and it therefore has some pre-image in $(\mathcal{Q}, \Lambda)$ a HOD-pair captured below κ; we see then that the direct limit system that generates $\pi^{j(\Sigma)}_{\infty}(\lambda^{j(\mathcal{P}^*)})$ is embedded (by the identity) into the direct limit system that generates $\pi_\infty^\Sigma(\lambda^{\mathcal{P}^*})$, and therefore "$\leq$" follows.

Finally we have by elementarity that $j(\kappa)$ is $(<\lambda, \Delta_x^\lambda)$-strong in N. Using the methods of Theorem 37 in N then gives that $\pi_\infty^{j(\Sigma)}(\lambda^{j(\mathcal{P}^*)}) \geq \Delta_x^\lambda$. ⊣

The following is then immediate:

COROLLARY 40. $D(M^\#_{>\theta_{\aleph_2}}, \lambda) \models \Theta > \theta_{\aleph_2}$.

§6. Non-linear degrees of hyperstrongness. Let λ be inaccessible or a limit of inaccessible cardinals. Let $R \subset V_\lambda$ be a pre-wellorder. Some notation: for $r \in \mathrm{dom}(R)$ we let $\delta_R(r)$ be the R-rank of r, Δ_R is the strict supremum of all ranks. Define the following notion by induction:

$\kappa < \lambda$ is $(<\lambda, R)$-strong iff for all $r \in \mathrm{dom}(R)$ and all $\gamma < \lambda$ there exists an extender E on κ such that $V_\gamma \subset \mathrm{Ult}(V; E)$, $i_E(R) \subset R$, and κ remains $(<i_E(\lambda), i_E(R) \restriction \{s \in \mathrm{dom}(i_E(R)) : s i_E(R) r\})$-strong in $\mathrm{Ult}(V; E)$.

For brevity's sake we will sometimes refer to an extender with critical point κ such that κ remains $(<i_E(\lambda), i_E(R) \restriction \{s \in \mathrm{dom}(i_E(R)) : s i_E(R) r\})$-strong in $\mathrm{Ult}(V; E)$ as a $(<\lambda, r)$-strong extender. It will usually be clear from context relative to which relation R this is to be understood.

LEMMA 41. *Let λ be Woodin. Then for all pre-wellorders $R \subset V_\lambda$ such that for all but boundedly many measurable cardinals $\mu < \lambda$ all elementary embeddings $j : V \to M$ with critical point μ satisfy $j(R) = R \cap N$, there exists a $\kappa < \lambda$ that is $(<\lambda, R)$-strong.*

PROOF. Let $A \subset V_\lambda$ be the set of pairs (α, r) where $\alpha < \lambda$ and $r \in \mathrm{dom}(R)$ and α is $(<\lambda, R \restriction \{s \in \mathrm{dom}(R) : sRr\})$-strong. Let κ be an A-reflecting cardinal. We claim that κ is $(<\lambda, R)$-strong.

Note that without loss of generality we have $j(R) \subset R$ for any elementary embedding $j : V \to M$ with critical point κ. Hence it follows that $\delta_{j(R)}(r) \leq \delta_{j(R)}(j(r))$. We see then that κ being $(<\lambda, R \restriction \{s \in \mathrm{dom}(R) : sRr\})$-strong for all $r \in \mathrm{dom}(R)$ is sufficient. (This is most evident when R is of limit-type, but only a minor adjustment to the argument is needed if R has a maximal degree.)

We will use induction on R to show just that: let $r \in \mathrm{dom}(R)$ be such that κ is $(<\lambda, R \restriction \{s \in \mathrm{dom}(R) : sRr'\})$-strong for all $r'Rr$, we will show that κ is $(<\lambda, R \restriction \{s \in \mathrm{dom}(R) : sRr\})$-strong.

For that purpose fix some arbitrary $r' \in \mathrm{dom}(R)$ with $r'Rr$ and some $\alpha < \lambda$. Let $j : V \to M$ be some embedding with critical point κ such that $V_\gamma \subset M$ and $j(A) \cap V_\gamma = A \cap V_\gamma$. Without loss of generality we can assume that $r' \in V_\gamma$, so by assumption we have $(\kappa, r') \in A$ and hence $(\kappa, r') \in j(A)$, but that means that κ is $(<\lambda, j(R) \restriction \{s \in \mathrm{dom}(j(R)) : s j(R) r'\})$-strong as wanted. (Recall here that we can assume that $r' j(R) r j(R) j(r)$ so this does indeed witness the required instance of κ being $(<\lambda, R \restriction \{s \in \mathrm{dom}(R) : sRr\})$-strong.) ⊣

There is a sufficiently rich class of pre-wellorders satisfying the requirements of this lemma: let λ be inaccessible or a limit of inaccessibles, a relation R on V_λ is $<\lambda$-u.b. iff there is some $\alpha < \lambda$ and a sequence $\langle \dot{T}_\beta, \dot{U}_\beta : \beta < \lambda \rangle$ of $\mathrm{Col}(\omega, \alpha)$-names such that:

- $(\dot{T}^g_\beta, \dot{U}^g_\beta)$ is $<\beta$-complenting over $V[g]$ for all $g \subset \mathrm{Col}(\omega, \alpha)$ generic over V and all $\beta < \lambda$;
- $(s,r) \in R$ if and only if $x \in (p\left[T^g_\beta\right])^{V[g][h]}$ for all $g \subset \mathrm{Col}(\omega,\alpha)$ generic over V, all $h \subset \mathrm{Col}(\omega, \mathrm{tc}((s,r)))$ generic over $V[g]$, all reals x coding (s,r) in $V[g][h]$, and all but unboundedly many $\beta < \lambda$.

It is standard to check that a $<\lambda$-u.B. relation will satisfy the requirement of the lemma. (If (T,U) are absolutely complementing and $j : V \to M$ is elementary then $(j(T), j(U))$ are complementing and $p[T] = p[j(T)]$.)

Note though that u.B. relations introduce a problem into our arguments: previously, the relations on indiscernibles had the remarkable property that the rank of any element of the domain would never change by moving to an ultrapower. We do not see that this would necessarily hold for u.B. relations in general. Hence it is unclear that given a κ which is $(<\lambda, R)$-strong for some $<\lambda$-u.B. pre-wellorder, κ is necessarily $(<\lambda, \Delta_R)$-strong.

Remember that in the proof of Theorem 37 we had to take embeddings such that some κ had a higher degree of hyperstrongness than the eventual image of some ξ in the direct limit system. We could do so trivially because we had precise control over the degree of hyperstrongness and the image of ξ under the direct limit embedding can only go down by restricting the domain to some ultrapower. This would be different with just any u.B. pre-wellorder. Working in M we could take some r such that $\delta_R(r) \geq \pi_\infty^{\Sigma^{\mathcal{P}}_{\mathcal{P}(0)}}(\xi)$ but that does not necessarily guarantee that $\delta_{j(R)}(r) \geq \pi_\infty^{j(\Sigma^{\mathcal{P}}_{\mathcal{P}(0)})}(\xi)$. Fortunately, there is a remarkably simple way around this conundrum. The direct limit system itself represents a pre-wellorder and we can index the degree of hyperstrongness with it.

Let $(\mathcal{P}, \Sigma)$ be a HOD-pair. We will then write $R_{(\mathcal{P},\Sigma)}$ for the pre-wellorder induced by its directed system, that is its domain consists of pairs $((\mathcal{Q},\Lambda),\alpha)$ of Σ-tails $(\mathcal{Q},\Lambda)$ and $\alpha \in \mathrm{On} \cap \mathcal{Q}$, and $((\mathcal{Q}_0,\Lambda_0),\alpha_0) \leq_{R_{(\mathcal{P},\Sigma)}} ((\mathcal{Q}_1,\Lambda_1),\alpha_1)$ iff given coiteration embeddings $\pi_0 : (\mathcal{Q}_0,\Lambda_0) \to (\mathcal{Q}_2,\Lambda_2)$ and $\pi_1 : (\mathcal{Q}_1,\Lambda_1) \to (\mathcal{Q}_2,\Lambda_2)$ we have $\pi_0(\alpha_0) \leq \pi_1(\alpha_1)$.

Definition 42. By $M^{\#}_{\theta_\Theta}$ we refer to the least (ω_1+1) -iterable (Mitchell-Steel-)premouse $(M; \in, \vec{E}, F)$ where F is non-empty and $M || \mathrm{crit}(F)$ satisfies the sentence:

"There is some λ, limit of Woodin cardinals and $<\lambda$-strong cardinals, such that for all $<\lambda$-strong cardinals $\kappa_0 \in V_\lambda$, if $(\mathcal{P},\Sigma)$ is some HOD-pair captured at κ_0 there exist some $\kappa_0 < \kappa_1 < \lambda$ which is $(<\lambda, R_{(\mathcal{P},\Sigma)})$-strong".

Lemma 43. *Assume $M := M^{\#}_{\theta_\Theta}$ exists and has a good (ω_1+1)-iteration strategy. The pair (M,λ) is tractable where λ is M's distinguished limit of Woodin cardinals.*

Proof. The only non-trivial part here is rebuilding. But if N is some non-dropping iterate of M, $O := \mathbb{L}^N_\mu$ for some μ below the image of λ, and $(\mathcal{P},\Sigma)$ is captured at some strong cardinal κ over O then it is also captured over the background model: let T, U be trees that witness η-u.B.-ness for some $\eta < \lambda$ and let x be any set in M such that that there exists a Woodin cardinal δ in between $\mathrm{rank}(x)$ and η; then x is generic at δ for the extender algebra at δ, thus either $x \in p[T]$ or $x \in p[U]$; so T, U also witness η-u.B.-ness over N. So there exists a $({<}\lambda, R_{(P,\Sigma)})$-strong cardinal in N. As shown in the proof of Lemma 30 extenders witnessing this will make it onto the sequence of O. ⊣

THEOREM 44. *Let (M, λ) be a tractable pair. Let $(\mathcal{R}, \Pi)$ be a hod-pair captured at some ${<}\lambda$-strong κ_0, and let $\kappa_0 < \kappa_1 < \lambda$ be a $({<}\lambda, R_{(\mathcal{R},\Pi)})$-strong. Let $\mathcal{P}$ be the local* HOD*-limit at κ_1, let $\mathcal{P}^*$ be its* Lp*-closure (relative to the derived model) and $\mathcal{R}^*$ the image of $\mathcal{R}$ under the direct limit embedding. Then $\lambda^{\mathcal{P}} \notin \mathcal{R}^*$ and if $\lambda^{\mathcal{P}} = \mathrm{On} \cap \mathcal{R}^*$ then $\mathcal{P}^*$ has a fullness preserving strategy with branch condensation in $D(M, \lambda)$.*

PROOF. The idea here is the same as in the proof of Theorem 37: if $\mathcal{T}$ is a tree on $\mathcal{R}^*$, then we can lift onto $\mathcal{P}^*$ giving rise to some $\mathcal{Q}^*$. Given $\alpha < \lambda^{\mathcal{Q}}$ we get $\Sigma^{\mathcal{Q}}_{\mathcal{Q}(\alpha)}$ by realizing into an ultrapower by some extender with critical point κ_1 and of big enough degree. The degree needed is exactly the degree of the iterate of $\mathcal{R}^*$ that generated α together with α's preimage!

Define the iteration strategy on a stack of normal trees $\vec{\mathcal{S}}$ as follows: Split $\vec{\mathcal{S}}$ into $\vec{\mathcal{U}}$ and $\vec{\mathcal{T}}$ where $\vec{\mathcal{U}}$ is on $\mathcal{R}^*$ and $\vec{\mathcal{T}}$ is above the image of $\mathcal{R}^*$. Let $(\mathcal{Q}, \Sigma_{\mathcal{Q}})$ be the last model of $\vec{\mathcal{U}}$ as a tree on $\mathcal{R}^*$, and let $\mathcal{Q}^*$ be its last model as a tree on $\mathcal{P}^*$. Say the last essential component of $\mathcal{T}$ is based on the $\xi + 1$-th layer, where $\xi + 1 \in \mathcal{Q}$. Take some $({<}\lambda, ((\mathcal{Q}, \Sigma_{\mathcal{Q}}), \xi + 1))$-strong extender such that $\vec{\mathcal{T}}$ is in the ultrapower. Prove by induction (like in Theorem 37) that the ultrapower is closed under $\Sigma^{\mathcal{Q}^*}_{\mathcal{Q}^*(\alpha)}$ for all $\alpha \leq \xi$. Then define the branch of $\vec{\mathcal{R}}$ by looking for realizable branches as before.

Now if $\lambda^P \in \mathcal{R}^*$, then we could take some $((\mathcal{R}^*, \Sigma^{\mathcal{R}^*}), \lambda^{\mathcal{P}})$-strong extender and get $\mathcal{P}^*$'s iteration strategy inside of the ultrapower which like in the proof of Theorem 37 will out-iterate $j(\mathcal{P})$. Contradiction! ⊣

COROLLARY 45. *Assume $M := M^{\#}_{\theta_\Theta}$ exists and has a good $(\omega_1 + 1)$-iteration strategy. $D(M, \lambda) \models \Theta = \theta_\Theta$ where λ is as before.*

PROOF. First we'll have to see that $D(M, \lambda)$ models $\mathrm{AD}_{\mathbb{R}}$. But λ is a limit of ${<}\lambda$-strong cardinals, so it follows. Let $\alpha < \Theta$ then there is some HOD-pair $(\mathcal{Q}, \Lambda)$ that iterates above α. Some such HOD-pair must be captured at some strong cardinal in some iterate of M. By the theorem we then get some HOD-pair $(\mathcal{P}, \Sigma)$ such that $\lambda^{\mathcal{P}}$ iterates above α. So then $\Theta > \theta_\alpha$. QED! ⊣

QUESTION 2. *If our guess is correct and $D(M, \lambda)$ is minimal with $\Theta = \theta_\Theta$ then it should not satisfy* DC*. Does it?*

We will end this paper with a short list of open questions.

QUESTION 3. *Is $M^{\#}_{\theta_{\omega_2}}$ the least mouse whose derived model satisfies $\Theta \geq \theta_{\omega_2}$?*

QUESTION 4. *Let M be the sharp mouse for the statement "there exists λ a limit of Woodin cardinals and there exists some $\kappa < \lambda$ that is $({<}\lambda, \Delta^{\lambda}_{x})$-strong for every $x \in V_\kappa$". Assume that M has a good (ω_1+1)-iteration strategy. Can we uniquely assign a "derived model" to M that satisfies $\Theta > \theta_{\omega_2}$?*

QUESTION 5. *Let λ be inaccessible or a limit of inaccessibles. Let R be a ${<}\lambda$-u.B. pre-wellorder on V_λ. Let $\kappa < \lambda$ be a $({<}\lambda, R)$-strong cardinal. Is κ $({<}\lambda, \Delta_R)$-strong?*

QUESTION 6. *Is $M^{\#}_{\theta_\Theta}$ the least mouse whose derived model satisfies $\Theta = \theta_\Theta$?*

QUESTION 7. *Assume that $M^{\#}_{refl}$ exists and has a good $(\omega_1{+}1)$-iteration strategy. Can we uniquely assign a "derived model" to $M^{\#}_{refl}$ and what is its theory?*

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