" $\mathrm{AD}_{\mathbb{R}} + \Theta$ is regular" from choiceless patterns of singulars

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In a symmetric extension of the HOD of a universe in which all uncountable cardinals are singular, there exists a model containing all the reals and satisfying "$\mathrm{AD}_{\mathbb{R}} + \Theta$ is regular".

1 Introduction

In his PhD thesis Gitik proved that there can exist models of Zermelo-Fraenkel set theory in which all uncountable cardinals are singular. To do this he started from a model of ZFC in which there exist proper class many strongly compact cardinals setting a high upper bound for the consistency strength of this property. (See [Git80])

A few years later Arthur Apter proved a "local" version of Gitik's Theorem, i.e. he proved that given the consistency of $\mathrm{ZF} + \mathrm{AD}$ there exists a model in which every uncountable cardinal less than Θ, the supremum of the length of pre-wellorders on $\mathbb{R}$, is singular. (See [Apt85])

Ralf Schindler and his student Daniel Busche, using the core model induction of Woodin (see [SS]), have shown:

Theorem 1.1 (Busche-Schindler): *Assume $V \models$ "All uncountable cardinals are singular". There is some cardinal μ and some $X \subset \mathrm{On}$ s.t. $\mathrm{AD}^{L(\mathbb{R})}$ holds in $\mathrm{HOD}_X[g]$ for all $g \subset \mathrm{Col}(\omega, \mu)$ generic over V.*

See [Bus08] or respectively [BS09]. The proof actually gives that $\mathbb{R}^{\#}$ exists in $\mathrm{HOD}_X[g]$, thus establishing that Gitik's "global" theorem is in fact stronger than Apter's "local" version.

Frankly, it is not surprising that a "local" property like AD does not yield a "global" theorem. Ralf Schindler, in unpublished work, showed that in Gitik's model there is an inner model with proper class many Woodin cardinals. But even this "global" property falls short. In this paper we will proof:

Theorem 1.2: *Assume $V \models$ "All uncountable cardinals are singular". There is some cardinal μ and some $X \subset \mathrm{On}$ s.t. in $\mathrm{HOD}_X[g]$ where $g \subset \mathrm{Col}(\omega, \mu)$ is generic over V there exists some $\Gamma \subset \mathcal{P}(\mathbb{R}^{\mathrm{HOD}_X[g]})$ with $L(\Gamma, \mathbb{R}^{\mathrm{HOD}_X[g]}) \models$ " $\mathrm{AD}_{\mathbb{R}} + \Theta$ is regular".*

Schindler and Busche's proof was based on [Ste05]. Similarly, our proof, intitally, will look a lot like [Sar14]. To reach "$\mathrm{AD}_{\mathbb{R}} + \Theta$ is regular" we will need to use "j-condensation" introduced in [Sar15] (in modern parlance we would use the term "condensing set"). A example of a core model induction using "j-condensation" or "condensing sets" can be found in [Tra].

This paper is organized thusly: the first section will introduce all the machinery needed for the proof; the second section is dedicated to the construction of a "maximal model" of "$\mathrm{AD} + \Theta = \theta_0$"; the third section will show how to get the next set beyond that maximal model; the fourth section will talk about iterating this process in a sustainable fashion; the fifth section will then use condensing sets to finish the proof of the theorem; the sixth section is a small appendix in which we will talk about what problems we face when trying to apply the machinery of [STa] to our problem and how to apply the methods of this paper to some similar problems.

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2 Preliminaries

2.1 Solovay Sequence

Let M be a transitive model of $\mathrm{ZF} + \mathrm{AD}$. Θ^M refers to the supremum of pre-wellorders on $\mathbb{R}$ in M. We will refine this notation. Define by induction a sequence of ordinals $\langle \theta_\alpha : \alpha \leq \beta \rangle$:

- $\theta_0 := \sup\{\gamma | \exists f \in \mathrm{OD}^M : f : \mathbb{R} \twoheadrightarrow \gamma\}$;
- $\theta_{\alpha+1} := \sup\{\gamma | \exists f \in \mathrm{OD}^M_A : f : \mathbb{R} \twoheadrightarrow \gamma\}$ if there is $A \in (\mathcal{P}(\mathbb{R}))^M$ with Wadge-degree θ_α (i.e. iff $\theta_\alpha < \Theta$) otherwise $\alpha = \beta$;
- $\theta_\lambda = \sup\limits_{\alpha<\lambda} \theta_\alpha$ if λ limit.

The length of this so called Solovay sequence is a natural degree of consistency strength for models of determinancy. At least at low levels every θ corresponds to a strong cardinal below a limit of Woodin cardinals:

Theorem 2.1 (Woodin): *(a) The following theories are equiconsistent:*

- "$\mathrm{ZF} + \mathrm{AD}$";
- "$\mathrm{ZFC} + \exists\lambda : \lambda$ *is a limit of Woodin cardinals*".

(b) The following theories are equiconsistent:

- "$\mathrm{ZF} + \mathrm{AD} + \Theta > \theta_0$";

– " ZFC $+\exists\kappa,\lambda : \lambda$ *is a limit of Woodin cardinals* , κ *is* $<\lambda$*-strong*".

(c) The following theories are equiconsistent:

– " ZF $+$ $\mathrm{AD}_{\mathbb{R}}$ ";

– " ZFC $+\exists\lambda : \lambda$ *is limit of Woodin cardinals and* $<\lambda$*-strong cardinals*".

One notices that compared to large cardinals this gives a rather coarse hierarchy. AD by itself is equiconsistent with infinitely many Woodin cardinals, but $\mathrm{AD} + \Theta > \theta_0$ is already far stronger consistency wise than a proper class of Woodin cardinals. Determinacy theories strictly in between have so far not been extensively studied. (But see [Tra14])

Θ itself can exhibit large cardinal properties. In this paper we will produce models of " $\mathrm{AD}_{\mathbb{R}} + \Theta$ is regular". (In the HOD of such a model Θ will be an inaccessible limit of Woodins.) It has been shown that this theory is weaker than the existence of a limit λ of Woodin cardinals and $<\lambda$-strong cardinals with some $\kappa < \lambda$ which reflects the set of $<\lambda$-strong cardinals. (G. Sargsyan and Y. Zhu, unpublished.)

2.2 Iteration Strategies

We do except our readers to be familiar with the basic language of inner model theory (see [MS94] or the handbook [Ste10]).

We say a premouse M is of Lp-type iff there is a set a s.t. M can be construed as an a-premouse, it is sound above a and projects below a. If we have two Lp-type premice M, N over a fixed a which can be compared then we'll have $M \trianglelefteq N$ or $N \trianglelefteq M$. Thus all Lp-type premice of the right kind can be gathered into one structure usually referred to as $\mathrm{Lp}(a)$.

We will make a distinction between two different kind of premice. The essential difference being the presence of a canonical well-order.

Definition 2.2: Let X, R, A be sets.

(a) X is *self-wellordered* iff $J_1(X)$ contains a wellorder on X.

(b) A premouse M over (R, A) is an $\mathbb{R}$-premouse iff $M \models R = \mathbb{R}$ and $A \subseteq R$.

Definition 2.3: (a) Let Γ be a inductive-like, determined pointclass, a a set. $\mathrm{Lp}^{\Gamma}(a)$ is the union of Lp-type premice M over a s.t. all countable hulls of M have (ω_1, ω_1) iteration strategies as coded by sets in Γ.

(b) Let a be a set. $\mathrm{Lp}(a)$ is the union of Lp-type premice M over a s.t. all countable hulls have an OD in X (ω_1, ω_1)-iteration strategy for some set of ordinals X.

Remark: Comparisons in the case of (a) can be performed in $L[T, M, N]$ where T is the tree of a scale on an universal Γ-set. In the case of (b) we can work in HOD_X. In all contexts where we need to do this, we'll have $(\omega_1)^{\mathrm{HOD}_X} < \omega_1$.

Neither of these definitions is supposed to be applied in a ZFC-context. (a) obviously presupposes a determinacy context and we will only use (b) in the context of our choiceless home universe. In the course of the core model induction we will by necessity also work in a ZFC-context.

During this process we will maintain that iteration strategies are of highest caliber. This good breeding expresses itself in the form of condensation properties which we are now going to list.

Definition 2.4: Let $M, \bar{M}$ be premice.

(a) Let $\mathcal{T}, \bar{\mathcal{T}}$ normal iteration trees on M and $\bar{M}$ respectively. We say $\bar{\mathcal{T}}$ is a hull of $\mathcal{T}$ (as witnessed by $\langle \sigma, \langle \pi_\beta : \beta < \mathrm{lh}(\bar{\mathcal{T}}) \rangle \rangle$ iff:
   - $\sigma : \mathrm{dom}(\bar{\mathcal{T}}) \to \mathrm{dom}(\mathcal{T})$ is order preserving, $\sigma(0) = 0$;
   - $\deg^{\bar{\mathcal{T}}}(\beta) = \deg^{\mathcal{T}}(\sigma(\beta))$, $D^{\bar{\mathcal{T}}} \cap (\beta, \gamma]_{\bar{\mathcal{T}}} = \emptyset$ iff $D^{\mathcal{T}} \cap (\sigma(\beta), \sigma(\gamma)]_{\mathcal{T}}$ for all $\beta \leq_{\bar{\mathcal{T}}} \gamma$ in the domain of $\bar{\mathcal{T}}$;
   - $\pi_\beta : M_\beta^{\bar{\mathcal{T}}} \to M_{\sigma(\beta)}^{\mathcal{T}}$ is a weak $\deg^{\bar{\mathcal{T}}}(\beta)$-embedding;
   - $\pi_\gamma \circ i_{\beta,\gamma}^{\bar{\mathcal{T}}} = i_{\sigma(\beta),\sigma(\gamma)}^{\mathcal{T}} \circ \pi_\beta$ whenever $\beta \leq_{\bar{\mathcal{T}}} \gamma$ and $D^{\bar{\mathcal{T}}} \cap (\beta, \gamma]_{\bar{\mathcal{T}}} = \emptyset$;
   - let $\beta := \mathrm{pred}^{\bar{\mathcal{T}}}(\gamma + 1)$ then $\sigma(\beta) = \mathrm{pred}^{\mathcal{T}}(\sigma(\gamma + 1))$ and $\pi_{\gamma+1}([a, f]_{E^{\bar{\mathcal{T}}_\gamma}}) = [\pi_\gamma(a), \pi_\beta(f)]_{E^{\mathcal{T}}_{\sigma(\gamma)}}$.

(b) Let $\langle \bar{\mathcal{T}}_\beta : \beta < \bar{\alpha} \rangle$ and $\langle \mathcal{T}_\beta : \beta < \alpha \rangle$ be two stacks of normal trees on $\bar{M}$ and M respectively. We say $\langle \bar{\mathcal{T}}_\beta : \beta < \bar{\alpha} \rangle$ is a hull of $\langle \mathcal{T}_\beta : \beta < \alpha \rangle$ (as witnessed by $\langle \sigma, \langle \sigma_\beta : \beta < \bar{\alpha} \rangle, \langle \pi_\gamma^\beta : \beta < \bar{\alpha}, \gamma < \mathrm{lh}(\bar{\mathcal{T}}_\beta) \rangle \rangle \rangle$ iff:
   - $\sigma : \bar{\alpha} \to \alpha$ is order preserving, $\sigma(0) = 0$;
   - $\mathcal{T}_\beta$ is a hull of $\mathcal{T}_{\sigma(\beta)}$ as witnessed by $\langle \sigma_\beta, \langle \pi_\gamma^\beta : \gamma < \mathrm{lh}(\bar{\mathcal{T}}_\beta) \rangle$.

Definition 2.5: Let M be a premouse and Σ a (possibly partial) iteration strategy for it. We say Σ has hull condensation iff for all stacks of normal trees $\vec{\mathcal{T}}, \vec{\mathcal{S}}$ on M, if $\vec{\mathcal{T}}$ is by Σ and $\vec{\mathcal{S}}$ is a hull of $\vec{\mathcal{T}}$ then $\vec{\mathcal{S}}$, too, is by Σ.

Remark: Let M be a premouse and Σ a strategy with hull condensation. Let $\vec{\mathcal{T}}$ be a stack of normal trees on M by Σ, let $\vec{\mathcal{S}}$ be a hull of $\vec{\mathcal{T}}$ as witnessed by $\langle \sigma, \langle \sigma_\beta : \beta < \mathrm{lh}(\vec{\mathcal{S}}) \rangle, \langle \pi_\gamma^\beta : \beta < \mathrm{lh}(\vec{\mathcal{S}}), \gamma < \mathrm{lh}(\vec{\mathcal{S}}_\beta) \rangle \rangle$ then $\vec{\mathcal{S}}$ is by $\Sigma^{\pi_0^0}$.

Definition 2.6: Let M be a premouse and Σ a (possibly partial) iteration strategy for it. We say Σ has branch condensation iff it has hull condensation and if there are N, a Σ-iterate with iteration embedding $\pi : M \to N$, and $\mathcal{T}$ a stack of normal trees by Σ together with a cofinal branch b s.t. the branch embedding $i_b^{\mathcal{T}}$ exists and there exists $\tau : M_b^{\mathcal{T}} \to N$ with $\tau \circ i_b^{\mathcal{T}} = \pi$, then $b = \Sigma(\mathcal{T})$.

Let M be a premouse and Σ an iteration strategy on it. Let $\mathcal{T}$ be a tree on M by Σ with last model N. We then write $\Sigma_{\mathcal{T},N}$ for the induced strategy on N.

Definition 2.7: Let M be a premouse and Σ an iteration strategy on it.

(a) Σ is positional iff $\Sigma_{N,\mathcal{T}} = \Sigma_{N,\mathcal{S}}$ for all trees $\mathcal{T}, \mathcal{S}$ by Σ with last model N.

(b) Σ is pullback consistent iff $\Sigma_{N,\mathcal{T}}^{i^{\mathcal{T}}}$ agrees with Σ on the intersection of their domains for all trees $\mathcal{T}$ on M by Σ with last model N and iteration embedding $i^{\mathcal{T}} : M \to N$.

(c) Σ has the weak Dodd-Jensen property iff $i^{\mathcal{T}} = i^{\mathcal{S}}$ for all trees $\mathcal{T}, \mathcal{S}$ by Σ with last model N and iteration embeddings $i^{\mathcal{T}}, i^{\mathcal{S}}$.

If a strategy Σ is positional we can then justifiably write Σ_N for the induced embedding of any iterate N.

Remark: If M is a Lp-type mouse over a the iteration strategy Σ of M above a is unique. From this it is not hard to see that Σ will have all the listed condensation properties.

We will now state a very general form of "generic iterability".

Definition 2.8: Let $X \subset \mathrm{On}$. Let $M \in \mathrm{HOD}_X$ be a premouse, α an ordinal or On. We say M is generically (α, α)-iterable iff

(a) M has an OD_X (α, α)-iteration strategy Σ with hull condensation;

(b) there exists some first order formula φ and parameter $\vec{p} \in \mathrm{HOD}_X$ s.t $\varphi(\cdot, \mathbb{P}, \vec{p})$ defines an (α, α)-iteration strategy Σ^g with hull condensation for any g generic over V for a forcing notion $\mathbb{P}$ of size $<\alpha$.

We have included the requirement for Σ to have hull condensation out of pure convenience. In application we do not want to have to make explicit that Σ^g has hull condensation, and we do not see how it is implied abstractly by Σ having hull condensation. Also note that in (b) it is implied that $\mathbb{P}$ can be well-ordered. In this paper we will generally only apply this definition to Levy collapses.

In applications we will need a stronger notion. We will need to know that generic extensions of strategies are consistent across mutually generic extensions.

Definition 2.9: Let $X \subset \mathrm{On}$. Let $M \in \mathrm{HOD}_X$ be a premouse, α an ordinal or On. We say M is strongly generically (α, α)-iterable iff it is generically (α, α)-iterable and in addition for any g generic over V for a forcing notion of size $<\alpha$ and $h_0, h_1 \in V[g]$ s.t. both are generic over V for a forcing notion of size $<\alpha$ we have that Σ^{h_0} and Σ^{h_1} agree on the intersection of their domains where Σ^{h_i} are the extensions given by generic iterability.

Fortunately, in many cases there is no difference between these two notions.

Lemma 2.10: *Let M be* Lp*-type that is generically* $(\mathrm{On}, \mathrm{On})$*-iterable. Then M is strongly generically* $(\mathrm{On}, \mathrm{On})$*-iterable.*

Proof: Let h_0, h_1, g as in the definition. W.l.o.g. assume that g is generic for $\mathrm{Col}(\omega, \alpha)$, some α. We can find $g_0, g_1 \subset \mathrm{Col}(\omega, \alpha)$ generic over $V[h_0]$ and $V[h_1]$ respectively s.t.

$$V[h_0][g_0] = V[g] = V[h_1][g_1].$$

Now by homogeneity the restriction of Σ^g to $V[h_i]$ is definable over that model as the unique strategy of M in some $\mathrm{Col}(\omega, \alpha)$ generic extension. By uniqueness of iteration strategies on M we thus have $\Sigma^{h_i} = \Sigma^g \restriction V[h_i]$. Q.E.D. $\dashv$

Remark: All our core model operators and HOD pairs will be strongly generically $(\mathrm{On}, \mathrm{On})$-iterable.

For cases not covered by the above lemma we will also have a use for a stronger notion which lends itself to "reflection" arguments. Unfortunately, it depends on choice but it will still prove quite useful.

Definition 2.11 (ZFC): Let M be a countable premouse and let Σ be an (α, α)-iteration strategy with hull condensation for M (up to α). We say Σ strongly determines itself on generic extensions iff there exists a formula φ, a parameter $\vec{p}$ and a club class C s.t. for all $\beta \in C$ there exists a stationary set S_β on $\mathcal{P}_{\omega_1}(H_\beta)$ s.t. for all $X \in S_\beta$ we have $\vec{p}, M \in X$ and if $\pi : X \to H$ is the transitive collapse and $\mathbb{P} \in H$ is such that $H \models \mathrm{Card}(\mathbb{P}) < \pi(\alpha)$ and $g \subset \mathbb{P}$ is generic over H then $\pi(H_\alpha)[g]$ is closed under Σ and $\varphi(\cdot, \mathbb{P}, \pi(\vec{p}))$ defines $\Sigma \restriction \pi(H_\alpha)[g]$ over $H[g]$. (In case $\alpha = \mathrm{On}$ we set $\pi(H_\alpha) := H$.)

Definition 2.12 (ZFC): Let M be a premouse and let Σ be an (α, α)-iteration strategy with hull condensation for M (up to α). We say Σ determines itself on generic extensions iff for some positive ordinal β there exists a $\mathrm{Col}(\omega, \beta)$-name $\dot{\Sigma}$ and a parameter $\vec{p}$ s.t. $\Vdash_{\mathrm{Col}(\omega,\beta)}$ "$\check{\Sigma} \subseteq \dot{\Sigma}$ is a $(\check{\alpha}, \check{\alpha})$-iteration strategy with hull condensation for $\check{M}$ s.t. $\dot{\Sigma}$ determines itself on generic extensions up to $\check{\alpha}$ as witnessed by $\check{\vec{p}}$".

Remark: Note that this is not quite the same definition as in [STb] as that definition depends on $M_1^{\Sigma,\#}$ and that won't do for our purposes. As we will see we will need generic iterability for Σ to get $M^{\Sigma,\#}$ in the first place. It is not hard to see that our version is strictly weaker, so we will be able to use some results from that paper.

Lemma 2.13 (ZFC): *Let M be a premouse, let Σ be a (α, α)-iteration strategy with hull condensation for M which determines itself on generic extensions up to α, then M is strongly generically (α, α)-iterable.*

PROOF: Using homogeneity we can w.l.o.g. assume that $\beta = 1$ and $\Sigma := \dot{\Sigma}^g$ ($g \equiv 0$) strongly determines itself on generic extensions.

Let $\mathbb{P}$ be some partial order of size $<\alpha$. Let $\varphi, \vec{p}, C$ be as in the definition of "strongly determines itself on generic extensions". Let $\beta \in C$ be sufficiently big s.t. $\vec{p}, \mathbb{P} \in H_\beta$.

Now assume for a contradiction that some $p \in \mathbb{P}$ forces that $\varphi(\cdot, \mathbb{P}, \vec{p})$ does not define the wanted extension. Let now $X \in S_\beta$ be sufficiently elementary with $p \in X$. Let then $\pi : X \to H$ be the transitive collapse. Let $g \subset \pi(\mathbb{P})$ be generic over H with $\pi(p) \in g$.

We then have that $\varphi(\cdot, \pi(\mathbb{P}), \pi(\vec{p}))$ defines $\Sigma_H^g := \Sigma \restriction \pi(H_\alpha)[g]$. An easy absoluteness argument shows $H[g] \models$ "Σ_H^g has hull condensation". Contradiction!

(b) is immediate: if in the situation as above h_i are generic over H with $h_0, h_1 \in H[g]$ then $\Sigma_H^{h_i}$ are both restrictions of Σ and hence agree. $\dashv$

Remark 2.14: It is not hard to see that if $\dot{\Sigma}$ is forced to have branch condensation, the Dodd-Jensen property etc, then the generic extension has branch condensation, the Dodd-Jensen property etc.

2.3 Hybrid Mice

We will mainly follow [STb] here. For our purposes a potential hybrid premouse will be a an acceptable J-structure of the form $\mathcal{N} := (J_\alpha^{\vec{E},\vec{B}}(A); \in, \vec{E}, \vec{B}, E, B, M)$ s.t.

- $\vec{E}$ is a fine extender sequence as described in [MS94], E is an amenable code for a coherent extender or failing that, empty;
- $M \in \mathrm{tc}(A)$ is a premouse;
- $\vec{B}^\frown B$ codes a partial iteration strategy for M;
- for all $\beta < \alpha$ at least one of $\vec{E}_\beta$ and $\vec{B}_\eta$ is empty, also at least one of E and B is empty;
- for all $\beta \leq \alpha$ if $(\vec{B}^\frown B)_\beta \neq \emptyset$ then there exist $\eta, \xi < \beta$ s.t. $\beta = \eta + \xi$ and $\mathcal{N}||\eta = (J_\eta^{\vec{E},\vec{B}}(A); \vec{E} \restriction \eta, \vec{B} \restriction \eta, M) \models \mathrm{ZF}$ and there exists some iteration tree $\mathcal{T} \in \mathcal{N}||\eta$ that is unique with the following properties
  - the last normal component of $\mathcal{T}$ has limit length,
  - $\mathrm{lh}(\mathcal{T}) \leq \xi$,
  - $\mathcal{T}$ is according to the partial iteration strategy coded by $\vec{B} \restriction \eta$,
  - $\mathcal{T}$ is not in the domain of the partial iteration strategy coded by $\vec{B} \restriction \eta$,
  - $\mathcal{N}||\eta \models \varphi(\mathcal{T})$

  and there exists a cofinal well-founded branch b through $\mathcal{T}$ and $(\vec{B}^\frown B)_\beta = \{\eta+\zeta | \zeta \in [0, \xi)_{\mathcal{T}^\frown b}\}$;
- let $\eta < \alpha$ and assume some $\mathcal{T}$ satisifies all the above requirements in $\mathcal{N}||\eta$, if $\vec{B}_\eta = \emptyset$ then for all $\xi < \min\{\mathrm{lh}(\mathcal{T}), \alpha - \eta\}$ we have that $(\vec{B}^\frown B)_\xi \neq \emptyset$.

The formula φ determines our organization scheme, it is a formula of the language $\mathcal{L}_{hyb}$ which is the language of set theory expanded by symbols $\dot{A}, \dot{\mathcal{B}}, \dot{\mathcal{E}}, \dot{E}, \dot{B}, \dot{M}$ (and others we need for the definition of fine extender sequence, but we will supress such details here).

If M is a premouse and Σ is a partial iteration strategy for it, then a structure $\mathcal{N} := (J_\alpha^{\vec{E},\vec{B}}(A); \in, \vec{E}, \vec{B}, E, B, M)$ is called a potential φ-organized Σ-premouse (over A) iff the partial iteration strategy coded by $\vec{B}^\frown B$ agrees with Σ. As is standard we write $\mathcal{N}|\beta := (J_\beta^{\vec{E},\vec{B}}(A); \in, \vec{E} \restriction \beta, \vec{B} \restriction \beta, (\vec{E}^\frown E)_\beta, (\vec{B}^\frown B)_\beta, M)$ and $\mathcal{N}||\beta := (J_\beta^{\vec{E},\vec{B}}(A); \in, \vec{E} \restriction \beta, \vec{B} \restriction \beta, M)$ where $\beta \leq \alpha$. We call these $\mathcal{N}$'s initial segments.

It is shown in [STb] that potential φ-organized Σ-premice obey the usual laws of fine structure as long as Σ has hull condensation. As usual we say a potential φ-organized Σ-premouse is a φ-organized Σ-premouse iff all its initial segments are sound. On the other hand we will require additional terms in the definition of iterability.

Definition 2.15: Let α, β be transitive classes of ordinals. Let M be a premouse and let Σ be a (possibly partial) iteration strategy for M. We say a φ-organized Σ-premouse $\mathcal{N}$ is (α, β)-iterable iff it is (α, β)-iterable in the sense of [MS94] and all such iterates are φ-organized Σ-premice.

[STb] shows that being a φ-organized hybrid premouse is preserved under iterations. On the other hand it should be intuitively clear that in general there is no first order statement that defines being a Σ-premouse which is why we have to require it in the definition.

Remark: Let $\mathcal{M}$ be a φ-organized Σ-premouse. Let Λ be an iteration strategy for $\mathcal{M}$. Nothing stops us from defining φ^*-organized Λ-premice. These so-called layered hybrid premice are an essential component of any advanced core model induction.

For our purposes we will need to consider two different ways to organize hybrid premice. This is because it is rather difficult to pick trees over a not wellordered set. ([STb] actually considers three different ways, but for our purposes we can ignore the difference between "g-organized" and "g-Θ-organized".)

Definition 2.16: Let ψ be a ZFC-formula. Let φ_ψ be the formula with one free variable t in the language of hybrid premouse that corresponds to the following statement:

"$\dot{A}$ is self-wellordered, t is an iteration tree on $\dot{M}$, t is according to the iteration strategy Σ coded by $\dot{\mathcal{B}}$ but $\Sigma(t)$ is not defined, $\psi(t)$, and t is minimal in the canonical wellorder with these properties."

ψ here can be used to restrict the domain to some desired class of iteration trees which can be occasionally useful if, say, we only have a strategy for normal trees. We can mostly ignore this here. If $\psi \equiv t = t$ we will supress the subscript.

This can legitimately be called the "standard scheme". Unfortunately, it doesn't really handle hybrid premice over not self-wellordered sets very well. We will need to have hybrid premice that can satisfy AD, so we will need a better way to organize our hybrid premice. As it turns out φ-organized hybrid premice will be a necessary ingredient.

Definition 2.17: Let $\mathcal{M}$ be a premouse, let $n < \omega$ and a a self-wellordered set s.t. $\mathcal{M} \in \mathrm{tc}(a)$, Σ a $(\mathrm{Card}(a)^+, \mathrm{Card}(a)^+)$-iteration strategy for $\mathcal{M}$. We write $M_n^{\Sigma,\#}$ for the least sound above a, φ-organized Σ-hybrid premouse $M := (M; \in, \vec{E}, \vec{B}, F, \mathcal{M})$ s.t. for all $\pi : \bar{M} \to_{\Sigma_1} M$ countable, $\bar{M}$ has a OD in some set of ordinals (ω_1, ω_1)-iteration strategy as a φ-organized Σ^π-hybrid premouse, $F \neq \emptyset$ and $M || \mathrm{crit}(F) \models$ " there are n Woodin cardinals".

Consider $M_1^{\Sigma,\#}$: it can interpret Σ on generic extension of any iterate, and using the extender algebra it can make any sufficiently small set generic over some iterate. In a sense $M_1^{\Sigma,\#}$ together with an iteration strategy Λ presents a master code for Σ.

Lemma 2.18 (ZFC): *Let M be a* Lp*-type premouse. Let $\alpha > \mathrm{Card}(M)$ be a cardinal and let Σ be a (α, α)-iteration strategy for M. If $M_1^{\Sigma,\#}$ exists and is (α, α)-iterable then Σ determines itself on generic extensions below α.*

This follows from Lemma 3.29 in [STb]. The theorem also holds for the suitable pairs and HOD-pairs we will introduce later! The result tells us in essence that the existence of a hybrid $M_1^\#$ is a strong form of generic iterability. This motivates the following definition which will be our goldstandard for all structures appearing during our core model induction.

Definition 2.19: Let a be a set.

(a) $\mathrm{I}(a)$ is the union of all Lp-type premice M over a which have an $(\mathrm{On}, \mathrm{On})$-iteration strategy Σ, $M_1^{\Sigma,\#}$ exists and is $(\mathrm{On}, \mathrm{On})$-iterable.

(b) $\mathrm{Lp}^+(a)$ is the union of all Lp-type premice M over a s.t. $\bar{M} \trianglelefteq \mathrm{I}(\bar{a})$ for all countable $\pi : \bar{M} \to M$, $\bar{a} = \pi^{-1}(a)$.

Lemma 2.18 also allows us to build an alternate hybrid premouse closed under Σ by feeding in the right trees on $M_1^{\Sigma,\#}$ instead.

Definition 2.20: Let N be a transitive set. Let M be a φ-organized Σ hybrid premouse for some (possibly partial) iteration strategy Σ s.t. M satisfies the first order theory of $M_1^{\Sigma,\#}$. An iteration tree $\mathcal{T}$ on M is an attempt at making N generically generic iff:

- if $\alpha < \mathrm{lh}(\mathcal{T})$ is less than $\mathrm{On} \cap N$, then $E_\alpha^{\mathcal{T}}$ is the least total measure of $\mathcal{M}_\alpha^{\mathcal{T}}$;
- if $\alpha < \mathrm{lh}(\mathcal{T})$ is greater or equal than $\mathrm{On} \cap N$, then $E := E_\alpha^{\mathcal{T}}$ is the least total extender on the sequence of $\mathcal{M}_\alpha^{\mathcal{T}}$ s.t. some $p \in \mathrm{Col}(\omega, N)$ forces that the generic real coding N violates some axiom induced by E;
- if $\alpha < \mathrm{lh}(\mathcal{T})$ is greater or equal than $\mathrm{On} \cap N$ and there is no E as above, then $\mathrm{lh}(\mathcal{T}) = \alpha + 1$.

We will refrain from giving a full definiton of $\varphi_{(g,\psi)}$ here, but it says something akin to: "t is an attempt at making some carefully chosen initial segment N of myself generically generic, t is least such in length that is according to my internal strategy but is not in the domain of my internal strategy, I have enough ordinals to ensure that t is actually making N generically generic, none of my initial segments in between N and including myself fail to satisfy ψ."

We will ignore ψ here and will just talk about g-organized hybrid premice. The right choice of ψ (namely $\psi \equiv$ "Θ exists") is important for the scale analysis of $\mathrm{Lp}^\Sigma(\mathbb{R})$. Here, we only need to know that the scale analysis succeeds, not why it does so.

Note that technically a g-organized hybrid premouse is a Λ-hybrid premouse - Λ being the iteration strategy of $M_1^{\Sigma,\#}$. But it will actually end up being closed under Σ, so we will refer to them as g-organized Σ-premice.

Definition 2.21: (a) Let Γ be an inductive-like, determined pointclass, $\mathcal{M}$ be a countable premouse and Σ an (ω_1, ω_1)-iteration strategy for $\mathcal{M}$ with branch condensation that can be coded by a set in Γ. Assume that $M_1^{\Sigma,\#}$ exists and is (ω_1, ω_1)-iterable. Let a be a set s.t. $\mathcal{M} \in \mathrm{tc}(a)$. $\mathrm{Lp}^{\Gamma,\Sigma}(a)$ is the union of Lp-type g-organized Σ-premice M over a s.t. all countable hulls of M have (ω_1, ω_1) iteration strategies as coded by sets in Γ.

(b) Let $\mathcal{M}$ be a premouse and Σ an $(\mathrm{On}, \mathrm{On})$-iteration strategy with branch condensation. Assume that $M_1^{\Sigma,\#}$ exists and is $(\mathrm{On}, \mathrm{On})$-iterable. Let a be a self-wellordered set s.t. $\mathcal{M} \in \mathrm{tc}(a)$. $\mathrm{Lp}^\Sigma(a)$ is the union of Lp-type g-organized Σ-premice M over a s.t. all countable hulls $\pi : \bar{M} \to M$ have an OD in X (ω_1, ω_1)-iteration strategy as Σ^π-premice for some set of ordinals X.

Remark: We will always be able to assume that $M_1^{\Sigma,\#}$ exists in the case of (b) above.

It should be easy to see that notions of iterabilty from the previous subsection generalize to hybrid premice. We will make the following definition explicit.

Definition 2.22: Let $\mathcal{M}$ be a premouse that is generically $(\mathrm{On}, \mathrm{On})$-iterable as witnessed by Σ. Assume that $M_1^{\Sigma,\#}$ exists and is generically $(\mathrm{On}, \mathrm{On})$-iterable. Let a be a set s.t. $\mathcal{M} \in \mathrm{tc}(a)$.

(a) $\mathrm{I}^{\Sigma}(a)$ is the union of Lp-type g-organized Σ-premice M over a that are $(\mathrm{On}, \mathrm{On})$-iterable by Λ, $M^{\Lambda,\#}$ exists and is $(\mathrm{On}, \mathrm{On})$-iterable.

(b) $\mathrm{Lp}^{+,\Sigma}(a)$ is the union of Lp-type g-organized Σ-premice M over a s.t. $\bar{M} \trianglelefteq \mathrm{I}^{\Sigma^{\pi}}(\bar{a})$ for all countable elementary $\pi : \bar{M} \to M$, $\bar{a} = \pi^{-1}(a)$.

A concluding remark: by necessity $M^{\Sigma,\#}$ will always be φ-organized. In all other contexts where we can assume that $M_1^{\Sigma,\#}$ exists, hybrid premice will always be g-organized. Even if they are defined over self-wellordered sets. There is a good reason for this as we will see later in this section.

2.4 Suitability

For the duration of this subsection we will let Γ be an inductive-like determined pointclass, i.e. Γ is closed under real quantification, is not self-dual and has the scale property.

Definition 2.23: Let $\mathcal{P}$ be a premouse, $n \leq \omega$. We say $\mathcal{P}$ is n-Γ-suitable iff:

- $\langle \delta_k^{\mathcal{P}} : k \leq n \rangle$ is an exhaustive list of $\mathcal{P}$'s Woodin cardinals and limits thereof, $\mathcal{P} = (\mathrm{Lp}^{\Gamma})^{\omega}(\mathcal{P}||\delta_n^{\mathcal{P}})$;
- whenever η is a strong cutpoint and cardinal of $\mathcal{P}$ then $\mathrm{Lp}^{\Gamma}(\mathcal{P}||\eta) \trianglelefteq \mathcal{P}$;
- $\mathrm{Lp}^{\Gamma}(\mathcal{P}||\xi) \models$ "ξ is not Woodin" for all $\xi \neq \delta_k^{\mathcal{P}}$ for some $k \leq n$.

From now on, if $n = 0$, then we will supress it.

Definition 2.24: Let $\mathcal{P}$ be a n-Γ-suitable premouse. An iteration tree $\mathcal{T}$ on $\mathcal{P}$ that concentrates on some window $\left(\delta_{k-1}^{\mathcal{P}}, \delta_k^{\mathcal{P}}\right)$ for some $k \leq n$ ($\delta_{-1}^{\mathcal{P}} := 0$) is ($\Gamma$-)correctly guided iff for all limit $\alpha < \mathrm{lh}(\mathcal{T})$:

- there is some $Q \trianglelefteq \mathrm{Lp}^{\Gamma}(\mathcal{M}(\mathcal{T}))$ s.t. Q defines a failure of $\delta(\mathcal{T})$ to be Woodin, $Q \trianglelefteq M_{\alpha}^{\mathcal{T}}$ and $b := [0, \alpha]_{\mathcal{T}}$ is the unique branch s.t $Q = Q(b, \mathcal{T})$;
- $\mathrm{Lp}^{\Gamma}(\mathcal{T}) \models \delta(\mathcal{T})$ is Woodin but $i_{\alpha}^{\mathcal{T}}(\delta^{\mathcal{P}}) \neq \delta(\mathcal{T})$ then there exists some $\beta < \alpha$ s.t. $\mathcal{T}_{\geq \beta}$ can be considered an iteration on $M_{\beta}^{\mathcal{T}}$ above some γ s.t. $\rho_{\omega}(M_{\beta}^{\mathcal{T}}) \leq \gamma$ (we say $\mathcal{T}$ has a fatal drop at (α, γ)).

Definition 2.25: Let $\mathcal{P}$ be n-Γ-suitable. Let $\mathcal{T}$ on $\mathcal{P}$ be correctly guided. If $Lp^{\Gamma}(\mathcal{M}(\mathcal{T}))$ defines a failure of $\delta(\mathcal{T})$ to be Woodin or $\mathcal{T}$ has a fatal drop, then $\mathcal{T}$ is called short. Otherwise, we say $\mathcal{T}$ is maximal.

Definition 2.26: Let $\mathcal{P}$ be n-Γ-suitable, Σ a (potentially partial) iteration strategy for $\mathcal{P}$. We say Σ is (Γ)-fullness preserving iff:

- whenever $\mathcal{T}$ is a tree by Σ and $\mathcal{U}$ is a normal component with base $\mathcal{M}$ and $\delta_k^{\mathcal{P}}$ has an image $\delta_k^{\mathcal{M}}$ for some $k \leq n$ and $\mathcal{U}$ concentrates on $(\delta_{k-1}^{\mathcal{M}}, \delta_k^{\mathcal{M}})$, then $\mathcal{U}$ without its last branch (if it exists) is correctly guided;

- if $\alpha < \mathrm{lh}(\mathcal{T})$ is limit and the branch from 0 to α does not drop then $M_\alpha^{\mathcal{T}}$ is n-Γ-suitable;
- if $\mathcal{T}$ has a fatal drop at (α, γ) then $\mathcal{T}_{\geq\alpha}$ is by the unique iteration strategy of $M_\alpha^{\mathcal{T}}$ for extenders with critical point above γ, furthermore we require this strategy to be in Γ.

Definition 2.27: Let $\mathcal{P}$ be n-Γ-suitable. $(\mathcal{P}, \Sigma)$ is a n-Γ-suitable pair, iff Σ is an (ω_1, ω_1)-iteration strategy with branch condensation that is Γ-fullness preserving.

The envelope of Γ, abridged $\mathrm{Env}(\Gamma)$, is the set of all $A \subseteq \mathbb{R}$ s.t. for a Turing-cone of countable $\sigma \subset \mathbb{R}$ we have $A \cap \sigma \in C_\Gamma(\sigma)$. It is characterized by the following property: if there exists a Suslin cardinal κ bigger than the prewellordering ordinal of Γ, then each set in Γ has a scale all of which individual prewellorders are coded by sets in $\mathrm{Env}(\Gamma)$. ([Jac10], section 3.2)

If Γ is determined then, assuming $\mathrm{DC}_{\mathbb{R}}$, so is $\mathrm{Env}(\Gamma)$. ([Wil15])

It can be shown that, in general, a Γ-suitable pair $(\mathcal{P}, \Sigma)$ does not exist s.t. a code for Σ is in the envelope of Γ, as any such pair uniformizes the complement of a Γ-universal set. But, as it turns out we can approximate such pairs from within Γ.

Definition 2.28: Let $\mathcal{P}$ be Γ-suitable. We say $\mathcal{P}$ is short-tree iterable iff whenever $\langle \mathcal{T}_k : k \leq n \rangle$ is such that

- $\mathcal{T}_0$ is on $\mathcal{P}$, $\mathcal{T}_k$ is on $(\mathrm{Lp}^\Gamma)^\omega(\mathcal{M}(\mathcal{T}_{k-1}))$ for all $0 < k \leq n$;
- $\mathcal{T}_k$ is maximal for all $k \leq n$;

then there exists a cofinal wellfounded branch b through $\mathcal{T}_n$ with $\mathcal{M}_b^{\mathcal{T}_n} = (\mathrm{Lp}^\Gamma)^\omega(\mathcal{M}(\mathcal{T}_n))$ and for all short trees $\mathcal{U}$ on $\mathcal{M}_b^{\mathcal{T}_n}$ there exists a well-founded cofinal branch c s.t. $\mathcal{U}^\frown c$ is correctly guided. (Note that c is unique.)

We'll refer to a stack as above as a *maximal stack*.

Definition 2.29: Let $\mathcal{P}$ be Γ-suitable and short-tree iterable. We say $\mathcal{Q}$ is a pseudo-iterate of $\mathcal{P}$ (by $\langle \langle \mathcal{T}_l : k \leq n \rangle, \mathcal{U} \rangle$) iff there is some maximal stack $\langle \mathcal{T}_k : k \leq n \rangle$ and $\mathcal{P}$ is the last model of some $\mathcal{U}$ a correctly guided short tree on $(\mathrm{Lp}^\Gamma)^\omega(\mathcal{M}(\mathcal{T}_k))$.

Given a suitable pair $(\mathcal{P}, \Sigma)$, Σ is completely determined by how it moves a fixed cofinal subset of $\delta^{\mathcal{P}}$. The ordinals in such a set can be represented by term-relations.

Definition 2.30: Let $A \subset \mathcal{R}$, M a countable transitive model of a suitable fragment of ZFC and $\alpha \in M$ an ordinal. We say M weakly term-captures A at α iff there exists a $\mathrm{Col}(\omega, \alpha)$-term τ s.t. $A \cap M[g] = \tau^g$ for all $g \subset \mathrm{Col}(\omega, \alpha)$ generic over M. Write

$$\tau_A^M := \{(p, \sigma) | p \in \mathrm{Col}(\omega, \alpha), \sigma \in M^{\mathrm{Col}(\omega,\alpha)}, p \Vdash \sigma \in \tau\}.$$

This does not depend on the choice of τ!

Definition 2.31: Let $\mathcal{P}$ be Γ-suitable and short tree iterable. Let $A \in \mathrm{Env}(\Gamma)$. We say $\mathcal{P}$ is weakly A-iterable, iff $\mathcal{P}$ and all its non-dropping pseudo-iterates weakly term-capture A, for all maximal stacks $\langle \mathcal{T}_k : k \leq n \rangle$ there exists some cofinal well-founded branch b through $\mathcal{T}_n$ s.t $M_b^{\mathcal{T}_n} = (\mathrm{Lp}^\Gamma)^\omega(\mathcal{M}(\mathcal{T})) =: \mathcal{Q}$ and $i_b^{\mathcal{T}_n}(\tau_A^{\mathcal{P}}) = \tau_A^{\mathcal{Q}}$ and $\sigma(\tau_A^{\mathcal{Q}}) = \tau_A^{\mathcal{R}}$ for all non-dropping short tree iteration embeddings $\sigma : \mathcal{Q} \to \mathcal{R}$.

Remark: Note that if $\mathcal{P}$ is Γ-suitable, weakly A-iterable then for a pseudo iterate $\mathcal{Q}$ by $\vec{\mathcal{T}} := \langle\langle\mathcal{T}_k : k \leq n\rangle, \mathcal{U}\rangle$ all branches b_k through $\mathcal{T}_k$ that move $\tau_A^{\mathcal{P}}$ correctly, i.e. to $\tau_A^{\mathcal{M}}$, will agree up to $\gamma_A^{\mathcal{M}} := \sup(\mathrm{Sk}^{\mathcal{M}}(\{\tau_A\})$ where $\mathcal{M} := \mathcal{M}_{b_k}^{\mathcal{T}_k}$. Thus there is a canonical $\pi_A^{\vec{\mathcal{T}}} : \mathrm{Hull}^{\mathcal{P}}(\gamma_A^{\mathcal{P}} \cup \{\tau_A^{\mathcal{P}}\}) \to \mathrm{Hull}^{\mathcal{Q}}(\gamma_A^{\mathcal{Q}} \cup \{\tau_A^{\mathcal{Q}}\})$.

We write $\mathcal{H}_A^{\mathcal{P}}$ for $\mathrm{Hull}^{\mathcal{P}}(\gamma_A^{\mathcal{P}} \cup \{\tau_A^{\mathcal{P}}\})$. If $\mathfrak{A}$ is a finite set of sets of reals, we let $\mathcal{H}_{\mathfrak{A}}^{\mathcal{P}}$ be $\mathcal{H}_{\bigoplus\mathfrak{A}}^{\mathcal{P}}$.

Definition 2.32: Let $\mathcal{P}$ be Γ-suitable and short tree iterable. Let $A \in \mathrm{Env}(\Gamma)$. We say $\mathcal{P}$ is strongly A-iterable, iff it is weakly A-iterable and for all tuples $\langle\mathcal{Q}_0, \mathcal{Q}_1, \mathcal{R}, \vec{\mathcal{T}}_0, \vec{\mathcal{T}}_1, \vec{\mathcal{U}}_0, \vec{\mathcal{U}}_1\rangle$ s.t.

- $\mathcal{Q}_i$ is a pseudo-iterate of $\mathcal{P}$ by $\vec{\mathcal{T}}_i$;
- $\mathcal{R}$ is a pseudo-iterate of $\mathcal{Q}_i$ by $\vec{\mathcal{U}}_i$;

we have $\pi_A^{\vec{\mathcal{U}}_0} \circ \pi_A^{\vec{\mathcal{T}}_0} = \pi_A^{\vec{\mathcal{U}}_1} \circ \pi_A^{\vec{\mathcal{T}}_0}$.

Lemma 2.33 (Woodin): *Let Γ be a determined inductive like pointclass. Let $A \in \mathrm{Env}(\Gamma)$ as witnessed by z, i.e. $A \cap \sigma \in C_\Gamma(\sigma)$ for all $\sigma \geq_T z$, and assume that Γ-mouse capturing holds, i.e. $C_\Gamma(x) \subseteq \mathrm{Lp}^\Gamma(x)$ for all $x \in \mathbb{R}$. Then there exists some Γ-suitable $\mathcal{P}(z)$ over z that is strongly A-iterable.*

Proof: This is Theorem 5.4.8 in [SS]. ⊣

Corollary 2.34: *Assume that additionally $\mathrm{Env}(\Gamma) \neq \mathcal{P}(\mathbb{R})$, then there exists some suitable pair $(P(z), \Sigma)$.*

Proof: Using the above mentioned properties of the envelope we do get a self-justifying system consisting of sets in $\mathrm{Env}(\Gamma)$. Using the methods of section 5.4 in [SS] we can then get Σ as the unique iteration strategy that moves all term relations for set in that self-justifying system correctly. ⊣

Let us now assume that M is a model of determinacy and $M \models \Theta = \theta_0$. Let $\Gamma = (\Sigma_1^2)^M$ and $z \in \mathbb{R}^M$. We can now define a directed system $\mathcal{F}$:

- the elements of $\mathcal{F}$ are Γ-suitable premice $\mathcal{P}(z)$ together with a finite set $\mathfrak{A}$ of $\mathrm{OD}^M(z)$ sets of reals s.t. $\mathcal{P}(z)$ is strongly A-iterable for all $A \in \mathfrak{A}$;
- $(\mathcal{P}(z), \mathfrak{A}) \leq_{\mathcal{F}} (\mathcal{Q}(z), \mathfrak{B})$ iff $\mathcal{Q}(z)$ is a pseudo-iterate of $\mathcal{P}(z)$ and $\mathfrak{A} \subset \mathfrak{B}$;
- whenever $(\mathcal{P}(z), \mathfrak{A}) \leq_{\mathcal{F}} (\mathcal{Q}(z), \mathfrak{B})$ we let $\pi^{\mathcal{F}}_{((\mathcal{P}(z),\mathfrak{A}),(\mathcal{Q}(z),\mathfrak{B}))} : \mathcal{H}_{\mathfrak{A}}^{\mathcal{P}(z)} \to \mathcal{H}_{\mathfrak{B}}^{\mathcal{Q}(z)}$ be $\pi_{\mathfrak{A}}^{\vec{\mathcal{T}}}$ for any $\vec{\mathcal{T}}$ that witnesses that $\mathcal{Q}(z)$ is a pseudo-iterate of $\mathcal{P}(z)$.

Let $\mathcal{H}(z)$ be the direct limit over $\mathcal{F}$. It has a unique Woodin cardinal $\delta^{\mathcal{H}(z)}$.

Lemma 2.35 (Steel-Woodin, see [SW16]): *$\mathcal{H}(z)$ is well-founded. $\delta^{\mathcal{H}(z)} = \Theta$ and $\mathcal{H}(z)||\delta^{\mathcal{H}(z)} = \mathrm{HOD}_z \cap V_\Theta$.*

We will want to use the language of suitable premice in a ZFC-context also. We say $\mathcal{P}$ is (ZFC)-suitable iff all the properties of a Γ-suitable premouse hold but with every mention of $\mathrm{Lp}^\Gamma(\cdot)$ replaced by $\mathrm{I}(\cdot)$. We will allow (ZFC)-suitable pairs to be larger than countable and be more than just (ω_1, ω_1)-iterable. In fact, our pairs will determine themselves on generic extensions. Of course, our ZFC-suitable premice will be suitable in some determinacy model for some pointclass, but that pointclass might only exist in a generic extension.

2.5 HOD-mice

This will be a short review of the notions and terms and the associated background knowledge we will lean on heavily during the proof. An in-depth treatise on the subject of HOD-mice (below "$AD_{\mathbb{R}} + \Theta$ regular", which is sufficient for our needs) can be found in [Sar]. With few exceptions proofs for the theorems and lemmata listed here can be found in [Sar], the theorem header will indicate the appropriate theorem number.

The background theory for this section is $\mathrm{ZF} + \mathrm{AD}^+$. We will also assume that no Wadge initial segment of our universe generates a model of $\mathrm{AD}_{\mathbb{R}} + \Theta$ is regular.

A HOD-premouse $\mathcal{P}$ is a ZFC^--structure of the following form: let $\langle \delta_i : i \leq \lambda^{\mathcal{P}} \rangle$ be a complete, increasing listing of all $\mathcal{P}$-cardinals which are Woodin cardinals or limits of Woodin cardinals (inside $\mathcal{P}$); $\mathcal{P}$ has $\lambda^{\mathcal{P}}$ layers $\mathcal{P}(i)$, $\mathcal{P}(i)$ has exactly ω cardinals above δ_i; $\mathcal{P}(i)^- := \mathcal{P}(i')$ iff $i = i' + 1$, $\mathcal{P}(i)^- := \mathcal{P}||\delta_i$ if i is limit, and $\mathcal{P}(0)^- := \emptyset$; $\mathcal{P}(i)$ is a g-organized hybrid mice relative to some partial strategy $\bigoplus_{i'<i} \Sigma^{\mathcal{P}}_{i'}$ for $\mathcal{P}(i)^-$ for all $i \leq \lambda^{\mathcal{P}}$; furthermore we require that if i is a limit ordinal then $((\delta_i)^+)^{\mathcal{P}(i+1)} = ((\delta_i)^+)^{\mathcal{P}(i)}$.

We say $\mathcal{P} \trianglelefteq_{\mathrm{HOD}} \mathcal{Q}$ iff both $\mathcal{P}$ and $\mathcal{Q}$ are HOD-premice and there exists some $\alpha \leq \lambda^{\mathcal{Q}}$ s.t. $\mathcal{P} = \mathcal{Q}(\alpha)$; write $\mathcal{P} \triangleleft_{\mathrm{HOD}} \mathcal{Q}$ iff aditionally $\alpha < \lambda^{\mathcal{Q}}$.

Definition 2.36: A HOD-pair $(\mathcal{P}, \Sigma)$ is a pair s.t. $\mathcal{P}$ is a countable HOD-premouse and Σ is a (ω_1, ω_1)-iteration strategy with hull condensation s.t. $\Sigma_{\mathcal{Q}(\alpha), \mathcal{T}} \cap \mathcal{Q} = \Sigma^{\mathcal{Q}}_{\alpha}$ for all iteration trees $\mathcal{T}$ according to Σ with last model $\mathcal{Q}$ and all $\alpha \leq \lambda^{\mathcal{Q}}$. Here $\Sigma_{\mathcal{Q}(\alpha), \mathcal{T}}$ refers to the iteration strategy on $\mathcal{Q}(\alpha)$ induced by Σ via $\mathcal{T}$ and $\Sigma^{\mathcal{Q}}_{\alpha}$ is the partial iteration strategy on the $\mathcal{Q}$-sequence.

We will often confuse $\Sigma^{\mathcal{Q}}_{\alpha}$ and $\Sigma_{\mathcal{Q}(\alpha), \mathcal{T}}$ in cases where the latter does not depend on $\mathcal{T}$, i.e. if Σ is positional, and in that case we might also write $\Sigma_{\mathcal{Q}(\alpha)}$. By the terms of the definition no harm will come from this.

Remark 2.37 (ZF): The above definition actually makes perfect sense outside of a determinacy context. We will want to allow uncountable structures as well. We write HP_{γ} for the set of all pairs $(\mathcal{P}, \Sigma)$ where $\mathcal{P}$ is a HOD-mouse of size at most γ, and Σ is a (γ^+, γ^+)-iteration strategy with hull condensation that satisfies the above requirements.

In all relevant cases we will have that our HOD-pairs will trace back to a HOD-pair in a determinacy model that exists in $V^{\mathrm{Col}(\omega, \gamma)}$. This will allow us to make use of results of this subsection even outside of a determinacy context.

Definition 2.38: Let $(\mathcal{P}, \Sigma)$ be a HOD-pair and Γ a pointclass. Σ is Γ-fullness preserving iff for all $\mathcal{T}$ according to Σ with last model $\mathcal{Q}$ s.t there is no drop on the main branch, $(\mathrm{Lp}^{\Gamma, \Sigma_{\mathcal{Q}(\alpha)^-, \mathcal{T}}})(\mathcal{Q}||\beta) \subset \mathcal{Q}$ for all cutpoints β of $\mathcal{Q}$ and α minimal with $\beta \in \mathcal{Q}(\alpha)$.

For HOD-pairs fullness preservation together with branch condensation is a core property that we will strive to have in every situation involving HOD-mice. For one, pairs with these properties will have all the usual regularity properties.

Lemma 2.39 (Sargsyan, 2.42): *Let (P, Σ) be a* HOD*-pair s.t. Σ has branch condensation and is $\mathcal{P}(\mathbb{R})$-fullness preserving. Then Σ is pullback-consistent, positional and has the weak Dodd-Jensen property.*

Note that the requirement in ([Sar], 2.42) for Σ to be Suslin-co-Suslin is always fulfilled as shown in ([Sar], 5.9).

To every $(\mathcal{P}, \Sigma)$ a HOD-pair, we associate a pointclass $\Gamma(\mathcal{P}, \Sigma)$. $\Gamma(\mathcal{P}, \Sigma)$ is approximately the pointclass of sets of lesser Wadge-degree than some code of Σ but this in itself is not a coherent definition. (Different codes will be projective in each other but there is no way to know that they share the same Wadge rank).

The correct definition for a limit type HOD-pair $(\mathcal{P}, \Sigma)$ is: $A \in \Gamma(\mathcal{P}, \Sigma)$ iff there exists an iteration tree $\mathcal{T}$ according to Σ with last model $\mathcal{Q}$, there is no drop on the main branch, there exists $\alpha < \lambda^{\mathcal{Q}}$ and A is Wadge reducible to a code of $\Sigma_{\mathcal{Q}(\alpha),\mathcal{T}}$.

We will omit the definition of $\Gamma(\mathcal{P}, \Sigma)$ in case $\mathcal{P}$ is a successor type. It can be found in ([Sar],Page 131).

Let $(\mathcal{P}, \Sigma)$ be a HOD-pair. We say $(\mathcal{Q}, \Lambda)$ is a tail of $(\mathcal{P}, \Sigma)$ iff there is some iteration tree $\mathcal{T}$ on $\mathcal{P}$ by Σ with last model $\mathcal{Q}$, the main branch of $\mathcal{T}$ does not drop, and $\Lambda = \Sigma_{\mathcal{Q},\mathcal{T}}$.

Let $(\mathcal{P}, \Sigma), (\mathcal{Q}, \Lambda)$ be two HOD-pairs. We say comparison holds between $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ iff there exist normal iteration trees $\mathcal{T}$ on $\mathcal{P}$ and $\mathcal{U}$ on $\mathcal{Q}$ by Σ and Λ with last models $\mathcal{P}^*$ and $\mathcal{Q}^*$ respectively, and

- $\mathcal{P}^* \trianglelefteq \mathcal{Q}^*$ and $\Sigma_{\mathcal{P}^*,\mathcal{T}} = \Lambda_{\mathcal{P}^*,\mathcal{U}}$
- or $\mathcal{Q}^* \trianglelefteq \mathcal{P}^*$ and $\Lambda_{\mathcal{Q}^*,\mathcal{U}} = \Sigma_{\mathcal{Q}^*,\mathcal{T}}$.

Note that we cannot expect comparison to hold everytime, e.g. if $\mathrm{Lp}^{\Gamma(P,\Sigma)}(a) \neq \mathrm{Lp}^{\Gamma(\mathcal{Q},\Lambda)}(a)$ for some a.

Lemma 2.40 (Sargsyan, 5.10): *Let $(\mathcal{P}, \Sigma), (\mathcal{Q}, \Lambda)$ be two* HOD*-pairs s.t. both Σ and Λ have branch condensation and are $\mathcal{P}(\mathbb{R})$-fullness preserving. Then comparison holds.*

At this point we still owe the audience a proof that anything but the most basic HOD-pairs exists. The next Lemma, known as "generation of pointclasses" shows that every "full" Wadge Initial segment of our Universe is generated by a HOD-pair.

Theorem 2.41 (Sargsyan, 6.1): *Let $\Gamma := \{A \in \mathcal{P}(\mathbb{R})^M | ||A||_w < \theta\}$ where $\theta < \Theta$ is an element of the Solovay sequence. Then there exists some* HOD*-pair $(\mathcal{P}, \Sigma)$ s.t. Σ has branch condensation and is Γ-fullness preserving and $\Gamma(\mathcal{P}, \Sigma) = \Gamma$.*

We can see that a long Solovay sequence induces complicated HOD-pairs. The converse is also true.

Theorem 2.42 (Sargsyan, 5.21): *Let $(\mathcal{P}, \Sigma)$ be a HOD-pair s.t. Σ is $\mathcal{P}(\mathbb{R})$-fullness preserving and has branch condensation. Let*

$$\mathcal{D} := \langle \{(\mathcal{Q}, \Lambda) : (\mathcal{Q}, \Lambda) \text{ a } \Sigma\text{-iterate of } (\mathcal{P}, \Sigma)\}, \{\pi_{\mathcal{Q},\mathcal{R}} : \pi_{\mathcal{Q},\mathcal{R}} \text{ it.-emb. }\}\rangle$$

be the directed system of all Σ-iterates of $(\mathcal{P}, \Sigma)$ together with the iteration embeddings. (Recall that Σ is positional)

Let $\mathcal{H}$ be the direct limit. $\pi_{\mathcal{Q},\infty} : \mathcal{Q} \to \mathcal{H}$ the direct limit embedding. Then

$$HOD \cap V_{\theta_{\pi_{\mathcal{Q},\infty}(\alpha)}} = \mathcal{H}||\pi_{\mathcal{Q},\infty}(\delta_\alpha^{\mathcal{Q}})$$

for all $(\mathcal{Q},\Lambda)\in\mathcal{D}$ *and all* $\alpha\leq\lambda^{\mathcal{Q}}$.

The preceding theorem goes under the label "HOD-analysis" (it also justifies the term HOD-mice as HOD is in fact a HOD-premouse, more or less). It has been proven for many different models at this point. The proof is always quite the same, and is essemtially the proof of Lemma 2.35. The crux being that it depends on "mouse capturing" which we do not know how to prove in general. Sargsyan has shown that it holds in the minimal model of $\mathrm{AD}_{\mathbb{R}}+\Theta$ regular and below.

Theorem 2.43 (Sargsyan, 6.19): *Let* $(\mathcal{P},\Sigma)$ *be a* HOD*-pair s.t.* Σ *is* $\Gamma(\mathcal{P},\Sigma)$*-fullness preserving and has branch condensation. Let* $x,y\in\mathbb{R}$ *and assume that* $y\in\mathrm{OD}(x,\Sigma)$ *then* $y\in\mathrm{Lp}^{\Sigma}(x)$.

This theorem has a counterpart for $\mathbb{R}$-premice.

Corollary 2.44 (Steel): *Let* $(\mathcal{P},\Sigma)$ *be a* HOD*-pair s.t.* Σ *is* $\Gamma(\mathcal{P},\Sigma)$*-fullness preserving and has branch condensation. Let* $A\subseteq\mathbb{R}$ *be* $\mathrm{OD}(\Sigma)$ *then* $A\in\mathrm{Lp}^{\Sigma}(\mathbb{R})$.

Proof: See [Ste] 17.1. ⊣

Note that the preceding HOD-analysis theorem cannot be used to analyse the "full" HOD as the requisite HOD-pairs cannot exist inside the model. We are left with the need for a more general concept of "suitable premouse". For our purposes we only need a rather neutered version.

Let $(\mathcal{P},\Sigma)$ be a HOD-pair s.t. Σ has branch condensation and is $\mathcal{P}(\mathbb{R})$-fullness preserving. Assume that the supremum of the length of OD_{Σ}-prewellorders on the reals is Θ, we say $\Theta=\theta_{\Sigma}$. It follows that any set of reals is $\mathrm{OD}_{\Sigma}(x)$ for some $x\in\mathbb{R}$.

We can now define a notion of Γ-suitable Σ-premice exactly as earlier but as a Σ-premouse. Let then $\Gamma=\Sigma^2_1(\Sigma)$ and $z\in\mathbb{R}$. We can now define a directed system $\mathcal{F}$:

- the elements of $\mathcal{F}$ are Γ-suitable Σ-premice $\mathcal{R}(z)$ together with a finite set $\mathfrak{A}$ of $\mathrm{OD}_{\Sigma}(z)$ sets of reals s.t. $\mathcal{R}(z)$ is strongly A-iterable for all $A\in\mathfrak{A}$;
- $(\mathcal{R}(z),\mathfrak{A})\leq_{\mathcal{F}}(\mathcal{Q}(z),\mathfrak{B})$ iff $\mathcal{Q}(z)$ is a pseudo-iterate of $\mathcal{R}(z)$ and $\mathfrak{A}\subset\mathfrak{B}$;
- whenever $(\mathcal{R}(z),\mathfrak{A})\leq_{\mathcal{F}}(\mathcal{Q}(z),\mathfrak{B})$ we let $\pi^{\mathcal{F}}_{((\mathcal{R}(z),\mathfrak{A}),(\mathcal{Q}(z),\mathfrak{B}))}:\mathcal{H}^{\mathcal{R}(z)}_{\mathfrak{A}}\to\mathcal{H}^{\mathcal{Q}(z)}_{\mathfrak{B}}$ be $\pi^{\vec{\mathcal{T}}}_{\mathfrak{A}}$ for any $\vec{\mathcal{T}}$ that witnesses that $\mathcal{Q}(z)$ is a pseudo-iterate of $\mathcal{R}(z)$.

Let $\mathcal{H}(z)$ be the direct limit over $\mathcal{F}$. It has a unique Woodin cardinal (above $\mathcal{P}$) $\delta^{\mathcal{H}(z)}$. We can relativize the arguments for Lemma 2.35 to get:

Lemma 2.45: $\mathcal{H}(z)$ *is well-founded.* $\delta^{\mathcal{H}(z)}=\Theta$ *and* $\mathcal{H}(z)||\delta^{\mathcal{H}(z)}=\mathrm{HOD}_{\Sigma,z}\cap V_{\Theta}$.

A vexing problem with branch condensation as compared to hull condensation is that it we cannot generally assume that a pullback of a strategy with branch condensation has branch condensation.

The next very useful lemma will show that this problem can be done away with when dealing with HOD-pairs by internalizing the property using the derived model.

Naturally, we will have to assume that we are dealing with limit types. This can be weakened slightly as seen here:

Lemma 2.46 (Sargsyan, 3.26): *Let $\langle(\mathcal{P}_\alpha, \Sigma_\alpha) : \alpha < \lambda\rangle$ be a sequence of* HOD*-pairs s.t.*

- *λ is a limit ordinal;*
- *$\mathcal{P}_\alpha \lhd_{\mathrm{HOD}} \mathcal{P}_\beta$ whenever $\alpha < \beta$, and Σ_α is the restriction of Σ_β to trees on $\mathcal{P}_\alpha$;*
- *Σ_α is $\bigcup_{\alpha<\lambda} \Gamma(\mathcal{P}_\alpha, \Sigma_\alpha)$ fullness preserving and has branch condensation for all $\alpha < \lambda$.*

Let $\mathcal{P} := \bigcup_{\alpha<\lambda} \mathcal{P}_\alpha$. Let $\pi : \bar{\mathcal{P}} \to \mathcal{P}$ elementary. Then whenever $\mathcal{P}_\alpha \in \mathrm{ran}(\pi)$ we have that $(\pi^{-1}(\mathcal{P}_\alpha), \Sigma^\pi_\alpha)$ is a HOD*-pair and Σ^π_α is $\bigcup_{\mathcal{P}_\alpha \in \mathrm{ran}(\pi)} \Gamma(\pi^{-1}(\mathcal{P}_\alpha), \Sigma^\pi_\alpha)$-fullness preserving and has branch condensation.*

2.6 S-constructions

Lemma 2.47 (S-construction lemma): *Let a be a set. $\mathbb{P} \in J_1(a)$ a partial order and $g \subset \mathbb{P}$ generic over* $\mathrm{Lp}(a)$*, then* $\mathrm{Lp}(a)\,[g] = \mathrm{Lp}(a\,[g])$*. The same holds for* $\mathrm{Lp}^\Gamma(a)$,$\mathrm{Lp}^+(a)$ *and* $\mathrm{I}(a)$.

Proof: For the " $\subseteq$ " direction we just need to note that the size of the forcing is small compared to the critical point of extenders on the sequence, thus extenders extend to the extension and this is also true for any iterates.

For the " $\supseteq$ " direction given a $a\,[g]$-premouse we can use the definability of the forcing relation to construct a premouse over a, preserving the fine-structure and iterability. We call this an S-construction here. (Originally this was called a P-construction, see [SS09]). ⊣

This is another occasion on which our prefernce for g-organized hybrid premice pays off.

Lemma 2.48 (S-construction lemma, hybrid version): *Let M be a fine structural model that is generically* $(\mathrm{On}, \mathrm{On})$*-iterable as witnessed by Σ. Assume $M_1^{\Sigma,\#}$ exists and is generically* $(\mathrm{On}, \mathrm{On})$*-iterable. Let a be a set s.t. $M \in \mathrm{tc}(a)$. $\mathbb{P} \in J_1(a)$ a partial order and $g \subset \mathbb{P}$ generic over* $\mathrm{Lp}^\Sigma(a)$*, then* $\mathrm{Lp}^\Sigma(a)\,[g] = \mathrm{Lp}^\Sigma(a\,[g])$*. The same holds for* $\mathrm{Lp}^{\Gamma,\Sigma}(a)$*,* $\mathrm{Lp}^{+,\Sigma}(a)$ *and* $\mathrm{I}^\Sigma(a)$.

Proof: Notice that when $N \trianglelefteq \mathrm{Lp}^\Sigma(a)$ models ZF then $\mathbb{P} \in N$ and it is absorbed into $\mathrm{Col}(\omega, N)$ over $\mathrm{Lp}^\Sigma(a)$. Hence the tree to make N generically generic and the tree to make $N\,[g]$ generically generic are the same. The rest is as above. ⊣

2.7 Vopenka Algebra

Core model theory, as far as we know, depends on the axiom of choice. Considering our background theory, this is a problem. We will deal with this by instead working in some inner model of choice, and if necessary extending operators and strategies to V by the use of the Vopenka Algebra.

The size of the Vopenka Algebra is the main reason why our arguments do not work in the Apter model.

Write $\Theta(\alpha)$ for $\sup\{\beta : \exists f : V_{\alpha+1} \twoheadrightarrow \beta\}$. We shall also set $\Theta(\omega) =: \Theta$ that way the notation is consistent with descriptive inner model theory.

Lemma 2.49: *Let $A, B \subset \mathrm{On}$ with $\mu := \sup(A)$. Then A is generic over HOD_B for a forcing notion of size $<\Theta(\mu+1)$ called the Vopenka algebra.*

PROOF: Let $f : \alpha \to (\mathcal{P}(\mathcal{P}(\mu))\backslash\{\emptyset\}) \cap \mathrm{OD}$ be an OD bijection. Then $\alpha < \Theta(\mu+1)$. Define $\mathbb{P} := (\alpha; \preceq)$ by $\beta \preceq \gamma$ iff $f(\beta) \subseteq f(\gamma)$. We then have that $G := \{\beta < \alpha | A \in f(\beta)\}$ is generic over HOD_B for $\mathbb{P}$. A can then be computed from G by $\xi \in A$ if and only if $f^{-1}(\{Y \subset \mu | \xi \in Y\}) \in G$. $\dashv$

Remark: Let κ be a cardinal that is Θ-closed, i.e. $\Theta(\alpha) < \kappa$ for all $\alpha < \kappa$. Let $X \subset \mathrm{On}$, then all bounded subsets of κ are generic over HOD_X for a $<\kappa$ size forcing notion. Note also that κ is a limit cardinal in V and a strong limit in HOD_X.

Lemma 2.50: *There exists a proper class of κ that is Θ-closed and for all $X \subset \mathrm{On}$ and all $\xi < \kappa$ there exists some $\mu < \kappa$ that is ξ-closed in HOD_X, i.e. $\mathrm{Card}(\mu^\xi) = \mu$.*

PROOF: Let κ be a Θ-closed ordinal s.t. the set of Θ-closed ordinals below κ has order-type κ. Obviously, the set of such κ is a proper class. Fix X, ξ as above. Let μ be the $(\xi^+)^{\mathrm{HOD}_X}$-th element in the enumeration of Θ-closed ordinals. By choice of κ we have $\mu < \kappa$.

Note that μ is both a strong limit in HOD_X and that its cofinality equals ξ^+ in HOD_X as the enumeration of Θ-closed ordinals is OD. We then have

$$\mu^\xi = \mu \cdot \sup_{\eta<\mu} \eta^\xi \leq \mu$$

as computed in HOD_X. Thus μ is as desired. $\dashv$

Let now κ_i be the i-th such κ. For the rest of the paper we will write $\kappa := \sup_{n<\omega} \kappa_n$.

2.8 Core model induction

At its core, "Core model induction" is the following process:

(1) Γ is a determined pointclass;

(2) identify Γ^+ the "next" pointclass with the scale property;

(3) show that Γ^+ is determined;

(4) repeat.

For example, let $\Gamma \subset L(\mathbb{R})$ be a determined inductivelike pointclass. Assuming that $\Gamma \neq (\Sigma^2_1)^{L(\mathbb{R})}$ we can then use [Ste08c] to find some β and $n \geq 1$ s.t. $\Gamma \subsetneq \mathcal{P}(\mathbb{R}) \cap J_\beta(\mathbb{R})$ and $\Sigma_n^{J_\beta(\mathbb{R})}$ has the scale property.

By Corollary 2.34 we then have some Γ-suitable pair (P, Σ). At this point in a core model induction we would usually leverage our hypothesis into extending Σ. Let us charitably assume that Σ can be uniquely extended to a $(\mathrm{On}, \mathrm{On})$-iteration strategy with branch condensation.

The way to reach the next determined pointclass is by the use of core model theory relativized to the theory of φ-organized (!) Σ-hybrid mice. For this we will have to momentarily assume that we work in an universe with choice.

Definition 2.51: Let Σ be a $(\mathrm{On},\mathrm{On})$-strategy for the finestructural model x with hull condensation. Let y be self-wellordered s.t. $x \in \mathrm{tc}(y)$. A $K^{c,\Sigma}$-construction over y is a sequence $\langle \mathcal{N}_\xi : \xi \leq \theta\rangle$ of φ-organized Σ-premice such that

(a) $\mathcal{N}_0 = (\mathrm{tc}(y); \in, \emptyset, \emptyset, x)$;

(b) if $\xi < \theta$ then $\mathcal{N}_\xi$ is solid and we let $\mathcal{C}_\omega(\mathcal{N}_\xi) =: \mathcal{M} = (M; \in, \vec{E}, \vec{B}, F, B, x)$, then:

  (i) either $\mathcal{M}$ is passive and $\mathcal{N}_{\xi+1} := (M; \in, \vec{E}, \vec{B}, E, \emptyset, x)$, where E is some extender cohering with $\mathcal{M}$, which is certified in the sense of [Ste96],

  (ii) or $\mathcal{N}_{\xi+1} := (M'; \in, \vec{E}, \vec{B}, \emptyset, B, x)$, where B is given by the definition of hybrid premice;

(c) if $\lambda \leq \theta$ is a limit, then $\mathcal{N}_\lambda = (N_\lambda; \in, (E^\alpha_\lambda : \alpha \in \mathrm{dom}(\vec{E}_\lambda)), (B^\alpha_\lambda : \alpha \in \mathrm{dom}(\vec{B}_\lambda)), \emptyset, \emptyset, x)$, where $\alpha \in \mathrm{dom}(\vec{E}_\lambda)$ iff $\alpha \in \mathrm{dom}(\vec{E}_\eta)$ for all but boundably many $\eta < \lambda$ and the sequence of the E^α_η is eventually constant, E^α_λ is then this eventual value, $\vec{B}_\lambda$ is defined analogously.

A $K^{c,\Sigma}$-construction is maximal iff we always add extenders at all levels where we are allowed to do so by the definition. As usual, as long as some minor iterability condtions are met, maximal $K^{c,\Sigma}$-constructions are unique.

Lemma 2.52 ($K^{c,\Sigma}$ dichotomy,ZFC): *Let Σ be a $(\mathrm{On},\mathrm{On})$-strategy for the finestructural model x with hull condensation. Assume that $M_n^{\Sigma,\#}$ exists for all self-wellordered y s.t. $x \in \mathrm{tc}(y)$. Fix some such y. Then*

- *either $M_{n+1}^{\Sigma,\#}(y)$ exists,*
- *or the unique maximal $K^{c,\Sigma}$-construction over y never breaks down and is $(\mathrm{On},\mathrm{On})$-iterable.*

Proof: The proof of [SS] works here as well. The only thing left to check is that following the realizable branch strategy produces Σ-premice. But any iterate by this strategy can be embedded into one of the $\mathcal{N}_\xi$, which are Σ-premice, so by condensation the iterate is a Σ-premice as well. ⊣

As usual a maximal $(\mathrm{On},\mathrm{On})$-iterable $K^{c,\Sigma}$ construction can be refined into a core model with the usual covering properties (see [Ste96],[JS]). In a core model induction we would leverage our hypothesis to show that necessarily we will have to come down on the "either"-side of our dichotomy. As usual, this will give that sets projective in a code for Σ are determined, closing the circle. See [SS] for greater detail.

In our specific case, Busche and Schindler did already do a lot of our work in this regard.

Theorem 2.53 (Busche-Schindler): *Let A be a set of ordinals that code V_κ^{HOD} in some straightforward fashion. Let $X \subset \mathrm{Lp}(A)$ be cofinal and have ordertype ω. Let ζ be s.t. every subset of ω_1 Vopenka-generic over HOD_X is already generic over $V_\zeta^{\mathrm{HOD}_X}$. Let $\mu < \kappa$ be a HOD_X-cardinal that is ζ-closed in HOD_X. Let $g \subset \mathrm{Col}(\omega, \mu)$ be generic over V.*

Then $L(\mathbb{R}^{\mathrm{HOD}_X[g]}) \models \mathrm{AD}^+$.

For our purposes this will not quite be enough. We will need:
Theorem 2.54: *Let A be a set of ordinals that code V_κ^{HOD} in some straightforward fashion. Let $X \subset \mathrm{Lp}(A)$ be cofinal and have ordertype ω. Let ζ be s.t. every subset of ω_1 Vopenka-generic over HOD_X is already generic over $V_\zeta^{\mathrm{HOD}_X}$. Let $\mu < \kappa$ be a HOD_X-cardinal that is ζ-closed in HOD_X. Let $g \subset \mathrm{Col}(\omega, \mu)$ be generic over V.*

Then $\mathrm{Lp}^+(\mathbb{R}^{\mathrm{HOD}_X[g]}) \models \mathrm{AD}^+$.

The proof here is actually quite the same, so we will omit it. We only need to substitute the scale analysis of [Ste08a] and [Ste08b] for the scale analysis of [Ste08c]. But, of course we still need a hybrid version of this theorem.
Theorem 2.55: *Let $\mathcal{M}$ be a finestructual model that is generically $(\mathrm{On}, \mathrm{On})$-iterable as witnessed by Σ and other parameters all of which are OD_Y for some set of ordinals Y. Assume that $\eta := \mathrm{Card}(\mathcal{M}) < \kappa$. Assume that $M_1^{\Sigma,\#}$ exists and is generically $(\mathrm{On}, \mathrm{On})$-iterable.*

Let A be a set of ordinals that code $V_\kappa^{\mathrm{HOD}_Y}$ in some straightforward fashion. Let $X \subset \mathrm{Lp}^\Sigma(A)$ be cofinal and have ordertype ω. Let ζ be s.t. every subset of ω_1 Vopenka-generic over $\mathrm{HOD}_{X,Y}$ is already generic over $V_\zeta^{\mathrm{HOD}_{X,Y}}$. Let $\eta \leq \mu < \kappa$ be a $\mathrm{HOD}_{X,Y}$-cardinal that is ζ-closed in $\mathrm{HOD}_{X,Y}$. Let $g \subset \mathrm{Col}(\omega, \mu)$ be generic over V.

Then $\mathrm{Lp}^{+,\Sigma}(\mathbb{R}^{\mathrm{HOD}_{X,Y}[g]}, \Sigma^g \restriction H_{\omega_1}^{\mathrm{HOD}_{X,Y}[g]}) \models \mathrm{AD}^+$.

Here we need the scale analysis of [STb].

Our immediate priority now is how to proceed to the next pointclass from $\mathcal{P}(\mathbb{R}) \cap \mathrm{Lp}^+(\mathbb{R}^{\mathrm{HOD}_X[g]})$.

3 A maximal model of $\mathrm{AD}^+ + \Theta = \theta_0$

Let ζ be some cardinal s.t. for all $X \subset \mathrm{On}$ all subsets of ω_1 are generic over $V_\zeta^{HOD_X}$;
Lemma 3.1: *Let $A \subset \mathrm{On}$ be a OD_X set where $X \subset \mathrm{Lp}(A)$ is cofinal and of ordertype ω. Let η be sufficiently big, and let $Y \prec V_\eta^{\mathrm{HOD}_X}$ be closed under ζ-sequences with $\mathrm{Lp}(A) \in Y$. Let $\pi : M \to Y$ be the reversed transitive collapse of Y. Then $\mathrm{Lp}(\pi^{-1}(A)) \in M$.*

Proof: We want to show $\pi^{-1}(\mathrm{Lp}(A)) = \mathrm{Lp}(\pi^{-1}(A))$. Assume not! Clearly, $\pi^{-1}(\mathrm{Lp}(A)) \trianglelefteq \mathrm{Lp}(\pi^{-1}(A))$ so we have a missing Lp-type premouse $\mathcal{M}$ over $\pi^{-1}(A)$.
Claim 1: $\mathrm{Ult}(\mathcal{M}; \pi)$ *is countably iterable.*

Proof of Claim: Let $\bar{\mathcal{M}}$ be a countable hull of $\mathrm{Ult}(\mathcal{M}; \pi)$. $\bar{\mathcal{M}}$ is generic over M for a size $<\zeta$-forcing. So we have $\pi^* : M\left[\bar{\mathcal{M}}\right] \to V_\eta^{\mathrm{HOD}_X}\left[\bar{\mathcal{M}}\right]$ an extension of π which is countably closed in $\mathrm{HOD}_X\left[\bar{\mathcal{M}}\right]$. By absoluteness $\bar{\mathcal{M}}$ is a hull of $\mathrm{Ult}(\mathcal{M}; \pi) = \mathrm{Ult}(\mathcal{M}; \pi^*)$ in $\mathrm{HOD}_X\left[\bar{\mathcal{M}}\right]$ so $\bar{\mathcal{M}}$ is a countable hull of $\mathcal{M}$ and thus has an OD_Z, for some Z, (ω_1, ω_1)-iteration strategy. □

We have that π is cofinal in $\mathrm{Lp}(A)$ so then $\mathrm{Lp}(A) \lhd \mathrm{Ult}(M;\pi)$. Contradiction! $\dashv$

Note that this theorem relativizes to hybrid-premice as long as the strategy satisfies hull condensation.

Lemma 3.2: *Let $X \subset \mathrm{On}$. Let $M \in \mathrm{HOD}_X$ be a premouse and Σ an $(\mathrm{On},\mathrm{On})$-iteration strategy that witnesses strong generic iterability for M in HOD_X. Also assume that (M,Σ) is such that $M_1^{\Sigma,\#}$ can generically interpret Σ as in Lemma 2.18, e.g. M is* Lp-*type. Then $M_1^{\Sigma,\#}(A)$ exists for all sets of ordinals $A \in V$ and is $(\mathrm{On},\mathrm{On})$-iterable in V.*

Proof: First note that Σ extends to a strategy with hull condensation over V which we will also call Σ. Simply a tree $\mathcal{T}$ is by Σ if it is by the extension of Sigma to $\mathrm{HOD}_X[\mathcal{T}]$. The terms of strong generic iterability ensure that this is a coherent definition.

We'll first have to show that Σ-$\#$ exists for all sets of ordinals in HOD_X. Let us fix $A \in \mathrm{HOD}_X$. Let η be a cardinal as in Lemma 2.50 s.t. $\mathrm{Card}(A) < \eta$. Take some $f : \omega \to (\eta^+)^{L^\Sigma(A)}$ cofinal of ordertype ω. We will have that $\Sigma \restriction \mathrm{HOD}_X[f]$ is definable over that model, hence the fine structure theory of $L^\Sigma(A)$ works as normal in that model. Clearly, f witnesses a failure of covering. Hence, $A^{\Sigma,\#}$ exists and is in HOD_X by an easy absoluteness argument involving the Levy collapse.

The same argument shows that $\mathrm{HOD}_X[Y]$ is Σ-$\#$-closed whenever $Y \subset \mathrm{On}$.

Fix some $A \in \mathrm{HOD}_X$ again. Let us now assume for a contradiction that $M_1^{\Sigma,\#}(A)$ does not exist as a mouse in V. Then we also have that $\mathrm{HOD}_X \models M_1^{\Sigma,\#}$ does not exist because Σ then determines itself on generic extensions over HOD_X and the universe is closed under Σ sharps. The usual absoluteness argument then gives that any local $M_1^{\Sigma,\#}(A)$ is actually iterable over V.

We can thus construct $K^\Sigma(A)$ inside of HOD_X. Let $\eta > \mathrm{Card}(A)$ be a V-cardinal. Let $f : \omega \to \eta^* := (\eta^+)^{K^\Sigma(A)}$ cofinal. Let μ be a regular in HOD_X cardinal s.t. the Vopenka Algebra to add f has size $< \mu$. Now consider a modified maximal $K^{\Sigma,c}$-construction $K_\mu^{\Sigma,c}$ s.t. all extenders used in the construction derive from hulls that are closed under μ-sequences.

This construction will still be certified in $\mathrm{HOD}_X[f]$. Therefore by the usual iterability proof it is still countably iterable there, and by the closure of $\mathrm{HOD}_X[f]$ under sharps it is actually $(\mathrm{On},\mathrm{On})$-iterable. Let $\nu > \mu$ be regular in HOD_X and sufficiently large. We can run the proof of stacking mice ([JSSS09]) in $\mathrm{HOD}_X^{\mathrm{Col}(\omega,\mu)}$ using a continuous sequences of substructures s.t. stationarily often the restriction of those structures to HOD_X is in HOD_X and closed under μ-sequences. Hence, $\mathrm{cof}(S^{\mathrm{HOD}_X^{\mathrm{Col}(\omega,\mu)}}(K_\mu^{\Sigma,c}(A)||\nu)) \geq \nu$.

We thus have that the above structure is an universal weasel in both HOD_X and $\mathrm{HOD}_X[f]$, and as ν can be arbitrarily large they must construct the same $K^\Sigma(A)$. But then f witnesses a failure of covering in $\mathrm{HOD}_X[f]$. So, $M^{\Sigma,\#}$ exists there and using generic interpretability it must be fully iterable in V. Contradiction! $\dashv$

Let now $A \subset \kappa$ be a set of ordinals that codes V_κ^{HOD} in a straightforward manner. Let $X \subset \mathrm{Lp}(A)$ be cofinal of ordertype ω. Whenever $\mu < \kappa$ is Θ-closed and ζ-closed in HOD_X we call it good.

Let us now fix some large enough η. A substructure Y of $H_\eta^{\mathrm{HOD}_X}$ is called "good" iff $\mu \subset Y$, $\mathrm{Card}^{\mathrm{HOD}_X}(Y) = \mu$, $\mathrm{Lp}(A) \in Y$ and Y is closed under ζ sequences. By choice of μ the set of good structures is stationary. While we are fixing this η for the most part, there are club-many of them which is important for e.g. Definition 2.12.

Lemma 3.3: *$M_\omega^\#$ exists and is* On*-iterable.*

PROOF: Let μ be good, let $g \subset \mathrm{Col}(\omega, \mu)$ be generic over V. By Theorem 2.53 working in $\mathrm{HOD}_X[g]$ we have $L(\mathbb{R}\cap\mathrm{HOD}_X[g]) \models \mathrm{AD}$. Let $\Gamma = (\Sigma_1^2)^{L(\mathbb{R}\cap\mathrm{HOD}_X[g])}$. $(\mathbb{R}\cap\mathrm{HOD}_X[g])^\#$ exists, therefore $\left[\delta_\Gamma, \mathrm{On}\cap(\mathbb{R}\cap\mathrm{HOD}_X[g])^\#\right]$ functions as a weak gap in $\mathrm{Lp}(\mathbb{R})$. We therefore get a self-justifying system $\langle A_i : i < \omega\rangle$ Wadge cofinal in $L(\mathbb{R}\cap\mathrm{HOD}_X[g])$. See [Ste08b].

We can then get a stack of premice $\langle \mathcal{P}(n) : n < \omega\rangle$ and iteration strategies Σ_n s.t.

- $\mathcal{P}(n)$ is a n-Γ-suitable premouse, $\delta_m^{\mathcal{P}(n)} = \delta_m^{\mathcal{P}(m)}$ for all $m \leq n$;
- Σ_n is a Γ-fullness-preserving (ω_1, ω_1)-iteration strategy on $\mathcal{P}(n)$ in $\mathrm{HOD}_X[g]$;
- let $\mathcal{P} := \bigcup_{n<\omega} \mathcal{P}(n)$ and

$$\mathcal{P}^+ := \begin{cases} J_\alpha(\mathcal{P}) & \alpha \text{ minimal with } \rho_\omega(J_\alpha(\mathcal{P})) < \delta_\omega^{\mathcal{P}} \\ L(\mathcal{P}) & \text{no such } \alpha \text{ exists}\end{cases}$$

  where $\delta_\omega^{\mathcal{P}} := \sup_{n<\omega} \delta_n^{\mathcal{P}(n)}$, then Σ_n acts on all of $\mathcal{P}^+$.

It is not hard to see that we can get a sequence like this for all inductive like $\Gamma^* \subset (\Delta_1^2)^{L(\mathbb{R}\cap\mathrm{HOD}_X[g])}$. (If Φ is a strategy for a course mouse that captures an universal Γ^*-set, then the full background construction of $M_1^{\Phi,\#}$ will reach such a sequence.)

Unfortunately, we can not Σ_1-reflect the existence of such pairs. But we can reflect for every A_i the existence of sequences of strongly A_i-iterable mice. A simultaneous comparison will then yield the required result.

For every $n < \omega$ we have that $\Sigma_n \restriction \mathrm{HOD}_X$ is a $((\mu^+)^{\mathrm{HOD}_X}, (\mu^+)^{\mathrm{HOD}_X})$-strategy.

CLAIM 1: $\bigoplus_{n<\omega} \Sigma_n$ *extends uniquely to a (normal tree) (κ, κ)-iteration strategy in* HOD_X.

PROOF OF CLAIM: This is just as in [Ste05] (Lemma 1.25). For the reader's convenience we will reproduce the argument here: given a good hull Y, $\pi : M \to Y$ the reverse of the Mostowski collapse, and $\mathcal{T} \in Y$ a tree, we write $\mathcal{T}_Y$ for the preimage of $\mathcal{T}$ under π and $b_Y^{\mathcal{T}} = \Sigma(\mathcal{T}_Y)$ if this is defined.

The extensions Σ^* is simply defined: $\mathcal{T}_Y$ is by Σ^* iff $\mathcal{T}_Y$ is by Σ for stationarily many good hulls Y. It is not hard to see that this defines Σ^* uniquely, we just have to show that it is total on trees of length less than κ in HOD_X.

We say a hull Y is stable (for $\mathcal{T}$) iff for all good $Z' \supseteq Z \supseteq Y$ we have that $\pi_{Z,Z'}$" $\left[b_Z^{\mathcal{T}}\right] \subseteq b_{Z'}^{\mathcal{T}}$ where $\pi_{Z,Z'}$ is the composition of the collapse of Z' with the reverse of the collapse of Z. We will show that there always is a stable Y. It is then not hard to see that we get some cofinal wellfounded branch b through $\mathcal{T}$ by Σ^*.

There are three cases: first assume that $\mathrm{cof}(\mathcal{T}) = \omega$. Because good hulls are countably closed, whenever Y is good with $\mathcal{T} \in Y$ and $\pi : M \to Y$ is the reverse of the Mostowksi collapse we have $b_Y \in M$. We can then find stationarily many hulls s.t. $\pi(b_Y)$ is constant. Any hull Y from this stationary set is stable. This is because whenever $Z \supseteq Y$ is good we can find $Z' \supseteq Z$ in our set. We skip further details.

Now let us assume that $\mathcal{T}_Y$ is short for stationarily many good Y. Fix such a Y, and let $\pi : M \to Y$ be the reverse of the Mostowski collapse. By Lemma 3.1 we have $\mathrm{Lp}(\bar{A}) \in M$. Note that $\mathrm{Lp}^\Gamma(\bar{A}) \subset \mathrm{Lp}(\bar{A})$. (This is because any countable in V hull of $\mathrm{Lp}^\Gamma(\bar{A})$ is $<\mu$-generic over HOD_X.) Hence the appropriate Q-structure for $\mathcal{T}_Y$ is in M. By a standard argument we then have $b_Y^{\mathcal{T}} \in M$. We can then press down on $\pi(b_Y^{\mathcal{T}})$. The rest is as in the previous case.

Let us now finally assume $\mathcal{T}_Y$ is maximal for stationarily many good Y. We can assume that $\mathrm{cof}(\mathcal{T}) > \omega$. Let us now fix two such hulls $Y \subseteq Z$. Let $c_{Y,Z}$ be the downward closure of $\pi_{Y,Z}\text{"}\left[b_Y^{\mathcal{T}}\right]$. Let $\eta := \sup c_{Y,Z}$. As $\mathrm{cof}(\eta) > \omega$ we have $c_{Y,Z} = [0, \eta)_{\mathcal{T}_Z}$. If $\eta = \mathrm{lh}(\mathcal{T}_Z)$ we are done. So assume not.

We will show that $\mathcal{T}_Z \restriction \eta$ is maximal. Therefore η will be a cutpoint in the tree and hence $c_{Y,Z} \subseteq b_Z^{\mathcal{T}}$ as wanted.

Because $\mathcal{T}_Y$ is maximal we have $i^{\mathcal{T}}_{b_Y^{\mathcal{T}}}(\delta_n) = \delta(\mathcal{T}_Y)$ for some n. Fix some $\rho < \delta_n$, we can then find some $\alpha \in b_Y^{\mathcal{T}}$ s.t. $\mathrm{crit}(i^{\mathcal{T}_Y}_{\alpha,\beta}) > i^{\mathcal{T}_Y}_{0,\alpha}(\rho)$ for all $\alpha < \beta \in b_Y^{\mathcal{T}}$. Lifting this pointwise under $\pi_{Y,Z}$ we get $i^{\mathcal{T}_Z}_{\pi_{Y,Z}(\alpha),\beta}(\rho) = i^{\mathcal{T}_Z}_{0,\pi_{Y,Z}(\alpha)}(\rho) < \delta(\mathcal{T}_Z \restriction \eta)$ for all $\pi_{Y,Z}(\alpha) < \beta \in c_{Y,Z}$, just as intended. $\square$

Σ^* which we will identify with Σ from now on then extends to generic extensions by the argument from [SZ08].

(For the reader's convenience we offer a sketch here: Let $h \subseteq \mathrm{Col}(\omega, \eta)$ $(\eta < \kappa)$ be generic over $V[g]$. Using the extended iteration strategy we get sets $A_i \subseteq A_i^* \subseteq \mathbb{R} \cap \mathrm{HOD}_X[g][h]$. (This argument will re-occur later in this paper). Now, with ω many Woodins we can use the extender algebra to internalize statements projective in the A_i^*. We then get

$$\langle H^{\mathrm{HOD}_X[g]}_{\omega_1}; \in, A_i : i < \omega \rangle \prec \langle H^{\mathrm{HOD}_X[g][h]}_{\omega_1}; \in, A_i^*; i < \omega \rangle.$$

Then "$\mathcal{P}(n)$ is strongly A_i^*-iterable" for all n, i gives rise to an extended iteration strategy.)

It is then easy to see that $\bigoplus_{n<\omega} \Sigma_n$ extends to a normal κ-iteration strategy.

Claim 2: *There is no $\alpha <$ On s.t. $\rho_\omega(J_\alpha(\mathcal{P})) < \sup\langle \delta^{\mathcal{P}(n)} : n < \omega \rangle$.*

Proof of Claim: Assume not. Let α be a minimal counterexample. Let m be minimal s.t. $\rho_\omega(\mathcal{P}^+) \leq \delta_m^{\mathcal{P}(m)}$. Let $\mathcal{Q}$ be the core of $\mathcal{P}^+$ above $\mathcal{P}(m)$. $\mathcal{Q}$ has a unique iteration strategy above $\mathcal{P}(m)$ and that is in $L(\mathbb{R} \cap \mathrm{HOD}_X[g])$. (It can be defined by the iteration strategy looking for a weakly iterable Q-structure. See [Ste10]).

The subset of $\mathcal{P}(m)$ that is defined by $\mathcal{Q}$, call it a, is thus in $\mathrm{HOD}^{L(\mathbb{R} \cap \mathrm{HOD}_X[g])}_{\mathcal{P}(m)}$. Hence, by mouse capturing, we have that $a \in \mathrm{Lp}^\Gamma(\mathcal{P}(m)) \subset \mathcal{P}$. Contradiction! $\square$

Thus $\mathcal{P}^{\#}$ is active and has ω Woodin cardinals. One can then show that this is κ-iterable. The next lemma will show that such a strategy extends to On. $\dashv$

Lemma 3.4: *Let M be some relativized premouse in HOD_X for some X. Say $\mathrm{Card}(M) = \alpha < \kappa$ and Σ is a (κ,κ)-iteration strategy with hull condensation that is OD_X. Then Σ extends (uniquely) to an $(\mathrm{On},\mathrm{On})$-iteration strategy with hull condensation.*

Proof: Uniqueness is easy, so we will leave that to the reader. Let $\mathcal{T}$ be some tree which is according to some partial extension with hull condensation. Let $Y \subset \mathrm{lh}(\mathcal{T})$ be cofinal of ordertype ω. Working in $\mathrm{HOD}_{X,Y}$ we get a countably stable hull, i.e. a countable hull $\mathcal{T}^*$ s.t. for every countable hull $\bar{\mathcal{T}}$ as witnessed by $\langle \sigma, \langle \pi_\beta : \beta < \mathrm{lh}(\bar{\mathcal{T}}) \rangle$ extending $\mathcal{T}^*$ and every other countable hull $\bar{\bar{\mathcal{T}}}$ of $\mathcal{T}$ as witnessed by $\langle \bar\sigma, \langle \bar\pi_\beta : \beta < \mathrm{lh}(\bar{\bar{\mathcal{T}}}) \rangle$ which extends $\bar{\mathcal{T}}$ as witnessed by $\langle \bar\sigma^{-1} \circ \sigma, \langle \bar\pi^{-1}_{\bar\sigma^{-1} \circ \sigma(\beta)} \circ \pi_\beta : \beta < \mathrm{lh}(\bar{\mathcal{T}}) \rangle$ we have $\bar\sigma^{-1} \circ \sigma\text{''} \left[\Sigma^{\pi_0}(\bar{\mathcal{T}})\right] \subseteq \Sigma^{\bar\pi_0}(\bar{\bar{\mathcal{T}}})$.

Assume for a contradiciton that there is no stable hull. We then have a sequence $\langle \bar{\mathcal{T}}_\alpha : \alpha < \omega_1^{\mathrm{HOD}_{X,Y}} \rangle$ s.t.

- $\bar{\mathcal{T}}_\alpha$ is a hull of $\mathcal{T}$ for all α as witnessed by $\langle \sigma_\alpha, \langle \pi^\alpha_\beta : \beta < \mathrm{lh}(\bar{\mathcal{T}}_\alpha) \rangle \rangle$;
- $\bar{\mathcal{T}}_{\alpha+1}$ extends $\bar{\mathcal{T}}_\alpha$ but $\sigma^{-1}_{\alpha+1} \circ \sigma_\alpha\text{''} \left[\Sigma^{\pi^\alpha_0}(\bar{\mathcal{T}}_\alpha)\right] \nsubseteq \Sigma^{\pi^{\alpha+1}_0}(\bar{\mathcal{T}}_{\alpha+1})$.

Let $\bar{\bar{\mathcal{T}}}$ be the direct limit and $\langle \sigma_*, \langle \pi^*_\beta : \beta < \mathrm{lh}(\bar{\bar{\mathcal{T}}}) \rangle \rangle$ the appropriate direct limit embeddings. Fix a cofinal subset a of $\Sigma^{\pi^*_0}(\bar{\bar{\mathcal{T}}})$. There exists some $\alpha < \omega_1^{\mathrm{HOD}_{X,Y}}$ s.t. $\sigma_*\text{''}\,[a]$ is covered by $\mathrm{ran}\,\sigma_\alpha$. Let b_α be the downward closure of $(\sigma_\alpha)^{-1} \circ \sigma_*\text{''}\,[a]$. It is then easy to see that $\bar{\mathcal{T}}_\alpha{}^\frown b_\alpha$ is a hull of $\bar{\bar{\mathcal{T}}}{}^\frown \Sigma^{\pi^*_0}(\bar{\bar{\mathcal{T}}})$. By hull condensation $b_\alpha = \Sigma^{\pi^\alpha_0}(\bar{\mathcal{T}}_\alpha)$. But the same is true for $\alpha+1$. So $\sigma^{-1}_{\alpha+1} \circ \sigma_\alpha\text{''} \left[\Sigma^{\pi^\alpha_0}(\bar{\mathcal{T}}_\alpha)\right] \subset \Sigma^{\pi^{\alpha+1}_0}(\bar{\mathcal{T}}_{\alpha+1})$ after all. Contradiction!

Fixing a stable hull $\bar{\mathcal{T}}$ as witnessed by $\langle \sigma, \langle \pi_\beta : \beta < \mathrm{lh}(\bar{\mathcal{T}} \rangle \rangle$ we can define a cofinal wellfounded branch b_Y through $\mathcal{T}$. $\xi \in b_Y$ iff there exists a hull $\bar{\bar{\mathcal{T}}}$ extending $\bar{\mathcal{T}}$ as witnessed by $\langle \bar\sigma, \langle \bar\pi_\beta : \beta < \mathrm{lh}(\bar{\bar{\mathcal{T}}}) \rangle \rangle$ s.t. $\bar\sigma^{-1}(\xi) \in \Sigma^{\bar\pi_0,}(\bar{\bar{\mathcal{T}}})$. This does not depend on the choice of stable hull but it might depend on Y. We have to eliminate that possibility.

Let us fix some $\pi : M \to V_\eta^{\mathrm{HOD}_{X,Y}}$ s.t. $\mathcal{T}^\frown b_Y \in \mathrm{ran}(\pi)$ and $\eta \subset M$ where β is such that every subset of M is $<\!\beta$-generic over $\mathrm{HOD}_{X,Y}$. Let $\bar{\mathcal{T}}^\frown b = \pi^{-1}(\mathcal{T}^\frown b_Y)$. By the construction of b_Y we have $\bar{\mathcal{T}}$ is by Σ and $b = \Sigma(\bar{\mathcal{T}})$.

We want to see that every countable hull $\bar{\bar{\mathcal{T}}}{}^\frown \bar b \in V$ of $\mathcal{T}^\frown b_Y$ as witnessed by $\langle \sigma, \langle \pi_\beta : \beta < \mathrm{lh}(\bar{\bar{\mathcal{T}}}) \rangle$ is by Σ^{π_0}. Let us fix some countable hull $\bar{\bar{\mathcal{T}}}{}^\frown \bar b$ as above then.

$(\bar{\bar{\mathcal{T}}}{}^\frown \bar b, \pi_0)$ is generic over $\mathrm{HOD}_{X,Y}$ for a $<\!\eta$ size forcing. π extends to

$$\pi^* : M\left[(\bar{\bar{\mathcal{T}}}{}^\frown \bar b, \pi_0)\right] \to V_\eta^{\mathrm{HOD}_{X,Y}}\left[(\bar{\bar{\mathcal{T}}}{}^\frown \bar b, \pi_0)\right].$$

By absoluteness $\bar{\bar{\mathcal{T}}}{}^\frown \bar b$ is a hull of $\mathcal{T}^\frown b_Y$, by elementarity it is then a hull of $\bar{\mathcal{T}}^\frown b$ as witnessed by $\langle \bar\sigma, \langle \bar\pi_\beta : \beta < \mathrm{lh}(\bar{\bar{\mathcal{T}}}) \rangle \rangle$ where $\bar\pi_0 = \pi_0$, but then $\bar{\bar{\mathcal{T}}}{}^\frown \bar b$ is by Σ^{π_0}.

We then now that b_Y does not depend on Y, because if there were an alternate b_Z we would have a hull $\bar{\mathcal{T}}^\frown b$ of $\mathcal{T}^\frown b_Y$ and a hull $\bar{\mathcal{T}}^\frown c$ of $\bar{\mathcal{T}}^\frown b_Z$ with $b \neq c$ but both hulls are according to some common pullback. Contradiction! $\dashv$

We only get a normal tree iteration strategy for $M_\omega^{\#}$ here. This will suffice for our purposes as we are only interested in genericity iterations. Note, though, that our core model induction will eventually reach a full iteration strategy for $M_\omega^{\#}$.

Lemma 3.5: *Let $X \subset \mathrm{On}$. Let $M \in \mathrm{HOD}_X$ be a premouse and Σ an $(\mathrm{On}, \mathrm{On})$-iteration strategy that witnesses strong generic iterability for M in HOD_X. Also assume that (M, Σ) is such that $M_1^{\Sigma,\#}$ can generically interpret Σ as in Lemma 2.18, e.g. M is* Lp*-type. Then $M_\omega^{\Sigma,\#}$ exists and is* On*-iterable.*

PROOF: The proof is essentially the same as the previous lemma, but instead we'll do our core model induction in $L(\mathbb{R} \cap \mathrm{HOD}_{Y,X'}[h], \Sigma \upharpoonright \mathbb{R} \cap \mathrm{HOD}_{Y,X'}[h])$. Here $X' \subset \mathrm{Lp}^{\Sigma}(A')$ cofinal and has ordertype ω, $h \subset \mathrm{Col}(\omega, \eta)$ generic over V, $\eta > \mathrm{Card}(M)$ is ζ-closed in $\mathrm{HOD}_{Y,X'}$, and A' codes $V_\kappa^{\mathrm{HOD}_Y}$ in some straightforward fashion. To even get started we'll need that $\Sigma \upharpoonright \mathbb{R} \cap \mathrm{HOD}_{Y,X'}[h]$ is self-scaled. Thankfully, we have already shown that $M_1^{\Sigma,\#}$ exists, so this is then provided by a result of [STb]. The necessary scale analysis can also be found in that paper. As before we get $M_\omega^{\Sigma,\#}$ in $\mathrm{HOD}_{Y,X'}[g]$. As $M_\omega^{\Sigma,\#}$ is definable, we do get it in HOD_Y too. $\dashv$

The above lemma shows that we cannot expect to go far by staying in one singular HOD_X. We will need to relate iterability between HODs the universe and generic extensions of either.

Lemma 3.6: *Let $X \subset \mathrm{On}$ a set.*

(a) Let $a \in \mathrm{HOD}_X$, then $\mathrm{I}^{\mathrm{HOD}_X}(a) = \mathrm{I}^V(a)$.

(b) Let $a \in \mathrm{HOD}_X$, α an ordinal, and $g \subset \mathrm{Col}(\omega, \alpha)$ generic over V, then $\mathrm{I}^{\mathrm{HOD}_X}(a) = \mathrm{I}^{\mathrm{HOD}_X[g]}(a)$.

(c) Let $\alpha \in \mathrm{On}$, $g \subset \mathrm{Col}(\omega, \alpha)$ and $a \in \mathrm{HOD}_X[g]$, then $\mathrm{I}^{\mathrm{HOD}_X[g]}(a) = \mathrm{I}^{V[g]}$.

PROOF: (a) "Left to right" is just Lemma 3.2, "right to left" uses the definability of iteration strategies for Lp-type premice and similarly for hybrid $M_1^{\#}$.

(b) "Left to right" utilizes the generic iterability given by Lemma 2.18 and the product lemma, "right to left" is homogeneity.

(c) "Left to right" is an adaptation of Lemma 3.2. We just have to realize that we can make subsets of ordinals in $V[g]$ generic over $\mathrm{HOD}_X[g]$ by making a $\mathrm{Col}(\omega, \alpha)$-name for such a set generic over HOD_X and using the product lemma. For "right to left" we just need to realize that the definability of iteration strategies for Lp-type premice means that the iteration strategy restricted to $\mathrm{HOD}_X[g]$ has a name in HOD_X. $\dashv$

For the sake of readability we will from now on omit relativizing $\mathrm{I}(\cdot)$ to models as above.

Remark: This combined with Lemma 3.4 gives us that generic (κ, κ)-iterability over HOD_X implies generic $(\mathrm{On}, \mathrm{On})$-iterability. (One must feel sorry for the intrepid thinkers of HOD_X as the true reason for this phenomenon is literally beyond their comprehension.)

We can now define the model with which we will work throughout the next section: we let $\Gamma_{\mu,g}^X$ be the downward closure of

$$\Gamma_{\mu,g}^X := \{A \subset \mathbb{R}^{\mathrm{HOD}_X[g]} | L(A, \mathbb{R}^{\mathrm{HOD}_X[g]}) \models \mathrm{AD}^+ + \Theta = \theta_0\}$$

under the Wadge order.

(NOTE: Write $\mathbb{R}$ for $\mathbb{R}^{\mathrm{HOD}_X[g]}$.)

Lemma 3.7: *Let $a \in \mathrm{HOD}_X[g]$ be countable and assume $M \trianglelefteq \mathrm{Lp}^{L(A,\mathbb{R})}(a)$ for some $A \in \Gamma^X_{\mu,g}$. Then M is strongly generically $(\mathrm{On}, \mathrm{On})$-iterable in $\mathrm{HOD}_X[g]$.*

Proof: Clearly, M has a (ω_1, ω_1) iteration strategy in $\mathrm{HOD}_X[g]$, call it Σ. Then there exists some inductive like scaled pointclass Γ s.t. $\Sigma \in \Gamma$. We can also assume that there is some further scaled pointclass beyond Σ. We can thus find some self justifying system $\langle A_i : i < \omega\rangle$ sealing the envelope of Γ. $\langle A_i : i < \omega\rangle$ will be definable over some real z. We want to assume that $\Sigma = A_0$.

By generic comparison of Γ-suitable z-premice we can then find some $\mathcal{P}(\tau) \in \mathrm{HOD}_X$ s.t. $\mathcal{P}(z) := \mathcal{P}(\tau)[g]$ is Γ-suitable and has a Γ-fullness preserving iteration strategy with branch condensation guided by $\langle A_i : i < \omega\rangle$, call it Σ^*. $\Sigma^* \restriction \mathrm{HOD}_X \in \mathrm{HOD}_X$ and is a (μ^+, μ^+)-iteration strategy there. Using the arguments from [Ste05] (Lemma 1.25) one can extend this to a (κ, κ)-iteration strategy over HOD_X.

(Here one uses that $\mathrm{Lp}^\Gamma(a) \trianglelefteq \mathrm{Lp}(a)$ for every $a \in \mathrm{HOD}_X$. Given a countable hull $M(\bar{a})$ somewhere in V, we can make this hull generic over HOD_X. Using both homogeneity and the absorption properties of the collapse, one gets

$$\Vdash_{\mathrm{Col}(\omega,\mu)} \check{M}(\bar{a}) \text{ has a } (\omega_1, \omega_1)\text{-iteration strategy.}$$

This argument can be found in [BS09].)

Using the ω-stack technique from [SZ08] (page 43 ff) one can show that this strategy determines itself on generic extensions. Using this and the fact that any iteration tree of lenth $<\kappa$ is $<\kappa$-generic over HOD_X we get that Σ^* extends to a (κ, κ)-iteration strategy with condensation over V and is OD_X (see proof of Lemma 3.2).

We want to show that M inherits this strategy. $\Sigma \restriction \mathcal{P}(z)|\delta$ can be defined over $\mathcal{P}(z)$. $\Sigma(\mathcal{T})$, $\mathcal{T} \in \mathcal{P}(z)|\delta$ is by Σ, is the unique branch b s.t. $\Vdash^{\mathcal{P}(z)}_{\mathrm{Col}(\omega,\delta)} (\dot{x}, \dot{y}) \in \tau^{\mathcal{P}(z)}_{A_0}$ where $\dot{x}, \dot{y}$ are names for reals that are forced to represent $\mathcal{T}$ and b respectively. The fact that there always exists such a b can be reflected to $\mathcal{P}(z)$ using genericity iterations.

Using this and the stationarity of background constructions one then shows that M iterates into the background construction. The background construction, of course, inherits an iteration strategy from $\mathcal{P}(z)$ which in turn is inherited by M. So M has a (κ, κ)-iteration strategy that determines itself on generic extensions. Lemma 3.4 takes care of the rest. ⊣

Lemma 3.8: *Let $a \in \mathrm{HOD}_X[g]$ be countable. Let M be an a-premouse of* Lp*-type that is strongly generically $(\mathrm{On}, \mathrm{On})$-iterable over $\mathrm{HOD}_X[g]$. Then there exists an $A \in \Gamma^X_{\mu,g}$ s.t. $L(A, \mathbb{R}) \models M$ is (ω_1, ω_1)-iterable .*

Proof: Let Σ be M's unique iteration strategy. By Lemma 3.5 applied in $V[g]$ we have that $M^{\Sigma,\#}_\omega$ exists. Thus $L(\mathbb{R}, \Sigma \restriction \mathbb{R}) \models \mathrm{AD}$. As Σ will be ordinal definable in that model it will also satisfy $\Theta = \theta_0$ as needed. ⊣

As an important corollary to the two preceding lemmata we have:

Corollary 3.9: *Let $a \in H^{\mathrm{HOD}_X[g]}$, then $\mathrm{Lp}^{\Gamma^X_{\mu,g}}(a) = \mathrm{I}^{\mathrm{HOD}_X[g]}(a)$.*

We can now assemble our maximal model. Let $S^X_{\mu,g} := L(\mathrm{Lp}^+(\mathbb{R}))$.

Lemma 3.10: $\Gamma^X_{\mu,g} = \mathcal{P}(\mathbb{R}) \cap S^X_{\mu,g}$.

Proof: If $A \in \Gamma^X_{\mu,g}$ then $L(A,\mathbb{R}) \models \mathrm{AD} + \Theta = \theta_0$, thus, by Corollary 2.44, A is contained in some $\mathbb{R}$-premice M all of whose countable hulls have (ω_1,ω_1)-iteration strategies inside of $L(A,\mathbb{R})$. By Lemma 3.7 $M \trianglelefteq \mathrm{Lp}^+(\mathbb{R})$.

On the other hand by Theorem 2.54 $S^X_{\mu,g} \models \mathrm{AD}^+$. We are done if we can show that $S^X_{\mu,g}$ believes "I am Lp($\mathbb{R}$)", because it then satisfies $\Theta = \theta_0$ and its sets of reals are therefore contained in $\Gamma^X_{\mu,g}$. So let $M \triangleleft \mathrm{Lp}^+(\mathbb{R})$ and let $\bar{M}$ be any countable hull. By definition $\bar{M}$ is generically $(\mathrm{On},\mathrm{On})$-iterable. By Lemma 3.8 we have that the iteration strategy for $\bar{M}$ is in $\Gamma^X_{\mu,g}$ and therefore in $S^X_{\mu,g}$. Thus $S^X_{\mu,g} \models M \trianglelefteq \mathrm{Lp}(\mathbb{R})$ as wanted. $\dashv$

4 A HOD pair for $S^X_{\mu,g}$

Lemma 4.1: *Let Y be good. Let M be its collapse. $\bar{A}, \bar{\kappa}$, etc the collapse of those things. Let $a \in V^M_{\bar{\kappa}}[g]$ then $\mathrm{I}^{M[g]}(a) = \mathrm{I}(a)$.*

Proof: We easily get $\mathrm{I}^M(a) \trianglelefteq \mathrm{I}(a)$ so we only need to take care of the other direction. Let $N \trianglelefteq \mathrm{I}(a)$ project to a. By Lemma 3.1 $\mathrm{Lp}(\bar{A}) \in M$. Clearly, $\mathrm{I}(\bar{A}) \trianglelefteq \mathrm{Lp}(\bar{A})$ so that too is in M. Let $\eta < \bar{\kappa}$ s.t. a has a name in V^M_η. Note that V^M_η is generic over the structure being coded by $\bar{A}$. Using S-constructions we can show that

$$\mathrm{I}(\bar{A})\left[V^M_\eta\right][g] = \mathrm{I}(\bar{A}\left[V^M_\eta\right][g]) \in M[g].$$

Let now

$$D := \{(x,y,\mathcal{T},b) | x \in \bar{A}\left[V^M_\eta\right][g], y \trianglelefteq \mathrm{I}(x), \rho_\omega(y) \leq x, \mathcal{T} \text{ is by } \Sigma_y, b = \Sigma_y(\mathcal{T})\}.$$

D is OD from $\bar{A}\left[V^M_\eta\right][g]$ which is countable in $\mathrm{HOD}_X[g]$. By mouse capturing - which we can apply here by Corollary 3.9 - $D \in \mathrm{I}(\bar{A}\left[V^M_\eta\right][g])$, thus $D \in M[g]$. From D we not only get $N \in M$ but we can easily define an (η,η)-iteration strategy on $M[g]$ for it. Now that strategy is unique so we can collect the iteration strategies we get for different η into a $(\bar{\kappa},\bar{\kappa})$-iteration strategy. Now the proof also works in $M[g][h]$ for every $\alpha < \bar{\kappa}$ and every $h \subset \mathrm{Col}(\omega,\alpha)$ generic over $M[g]$ with $h \in \mathrm{HOD}_X[g]$. So M believes that N is generically $(\bar{\kappa},\bar{\kappa})$-iterable, but it also believes that any such N is actually generically $(\mathrm{On},\mathrm{On})$-iterable. QED! $\dashv$

Remark: Note that whenever Y is good and M is its transitive collapse and h is generic over $M[g]$ for a $<\bar{\kappa}$-size forcing notion then $\mathrm{I}(\cdot) \restriction V^{M[g][h]}_{\bar{\kappa}} \in M[g][h]$. What we do *not* know is that $\mathrm{I}^{M[g][h]}(a) = \mathrm{I}(a)$ for all $a \in V^{M[g][h]}_{\bar{\kappa}}$ or even that it is definable over $M[g][h]$ from parameters in M.

Set $\Theta^X_{\mu,g} := \Theta^{S^X_{\mu,g}}$ and $\mathcal{P}^X_\mu := \mathrm{HOD}^{S^X_{\mu,g}} || (\Theta^X_{\mu,g})^{+\omega}$ (by homogeneity this is independent of g). Given a good hull Y and $\pi : M \to Y$ the reverse of its transitive collapse, write $\mathcal{P}^\pi$ for the preimage of $\mathcal{P}^X_\mu$ under π. For $a \in H^{\mathrm{HOD}_X}_{\mu^+}$ we might also consider $\mathcal{P}^X_\mu(a) := \mathrm{HOD}^{S^X_{\mu,g}}_a || (\Theta^X_{\mu,g})^{+\omega}$.

Lemma 4.2: *Let Y be a good hull. Let M be the transitive collapse, π the reverse collapse embedding. Let $a \in V_{\bar{\kappa}}^M$ and assume there is $b \in H_{\omega_1}^{\mathrm{HOD}_X[g]}$, $\sigma : \mathrm{I}^\omega(a) \to b$ in $\mathrm{HOD}_X[g]$ s.t. there is $\tau : b \to \pi(\mathrm{I}^\omega(a))$ with $\pi = \tau \circ \sigma$. Then $b = \mathrm{I}^\omega(\sigma(a))$.*

Proof: Let T be the tree of a scale on the universal Σ_1^2 set in $S_{\mu,g}^X$. By mouse capturing all bounded subsets of $\mathrm{On} \cap \mathrm{I}^n(a)$ in $L[T, \mathrm{I}^\omega(a)]$ are in $\mathrm{I}^{n+1}(a)$ which is in M by Lemma 4.1. So we can form the long extender ultrapower of $L[T, \mathrm{I}^\omega(a)]$ by σ which can be embedded into the long extender ultrapower of that model by $\pi \restriction \mathrm{I}^\omega(a)$. So it is wellfounded by countable completeness of π. Write σ^+ for the ultrapower embedding.

Let n be minimal s.t. $\sigma(\mathrm{I}^{n+1}(a))$ is missing a mouse, call it N. Fix $k < \omega$ s.t. $(T)_k$ projects to

$$\{(x,y,z) | x \text{ codes } c \in H_{\omega_1}^{\mathrm{HOD}_X[g]}, (y,z) \text{ code } (N', N'') : N' \lhd N'' \lhd \mathrm{I}(c)\}.$$

So we have $(x,y,z) \in p[(T)_k]$ for any real $x \in \mathrm{HOD}_X[g]$ coding $\sigma(\mathrm{I}^n(a))$, any real y coding $\sigma(I^{n+1}(a))$, and any real z coding N. By standard arguments the same then holds for $\sigma^+((T)_k)$.

On the other hand the following holds in any $\mathrm{Col}(\omega, \mathrm{I}^{n+1}(a))$-generic extension of $L[T, \mathrm{I}^\omega(a)]$: "for any real x coding $\mathrm{I}^n(a)$, for all reals y coding $\mathrm{I}^{n+1}(a)$ there is no real z with $(x,y,z) \in p[(T)_k]$."

So a corresponding statement holds in $L[\sigma^+(T), b]$. Now we can take in $\mathrm{HOD}_X[g]$ some $h \subset \mathrm{Col}(\omega, \sigma(\mathrm{I}^{n+1}(a)))$ generic over $L[\sigma^+(T), b]$; for any (x,y) coding $\sigma((\mathrm{I}^n(a), \mathrm{I}^{n+1}(a))$ there is then some real z s.t. $(x,y,z) \in p[(\sigma^+(T))_k]$ as witnessed by any z coding N.

In $L[\sigma^+(T), b][h]$ by absoluteness for any (x,y) coding $\sigma((\mathrm{I}^n(a), \mathrm{I}^{n+1}(a))$ there is then some z with $(x,y,z) \in p[(\sigma^+(T))_k]$ and z codes some structure end-extending $\sigma(\mathrm{I}^{n+1}(a))$. Contradiction! $\dashv$

Lemma 4.3: *Let Y be a good hull, $\pi : M \to Y$ be the reverse of the Mostowski collapse. $\mathcal{P}^\pi$ has a (ω_1, ω_1) Γ^X-fullness preserving iteration strategy Σ^π in $\mathrm{HOD}_X[g]$. Furthermore, $\Sigma^\pi \restriction \mathrm{HOD}_X \in \mathrm{HOD}_X$.*

Proof: Let $\langle \dot{B}_i : i < \mu \rangle$ be an exhaustive list of all $\mathrm{Col}(\omega, \mu)$-names that are forced to be sets of reals, ordinal definable in $S_{\mu,g}^X$. Let $\langle C_n : n < \omega \rangle = \langle \dot{B}_i^g : i < \mu \rangle$ in $\mathrm{HOD}_X[g]$. By elementarity $\mathcal{P}^\pi$ is a pseudo-iterate of a C_n-strongly iterable ($\Gamma_{\mu,g}^X$-)suitable premouse. Hence $\mathcal{P}^\pi$ is strongly C_n-iterable for all $n < \omega$.

Now, given a suitable premouse $\mathcal{Q}$, strongly C_k-iterable for all $k \leq n$, we let $\tau_k^{\mathcal{Q}}$ be the canonical term capturing C_k, let $\gamma_n^{\mathcal{Q}}$ be $\sup(\mathrm{Hull}^{\mathcal{Q}}(\tau_k^{\mathcal{Q}} : k \leq n) \cap \delta^{\mathcal{Q}})$ and let $\mathcal{H}_n^{\mathcal{Q}}$ be the transitive collapse of $\mathrm{Hull}^{\mathcal{Q}}(\gamma_n^{\mathcal{Q}} \cup \{\tau_k^{\mathcal{Q}} : k \leq n\})$.

Defining Σ on an appropriate $\mathcal{T}$, there are three possibilities:

If $\mathcal{T}$ has a fatal drop, i.e. $\mathcal{T}_{\geq \alpha}$ can be considered a tree on $\mathrm{I}(M)$ for some $\alpha < \mathrm{lh}(\mathcal{T})$ and cutpoint initals segment M of $\mathcal{M}_\alpha^{\mathcal{T}}$, then $\Sigma(\mathcal{T})$ is the branch given by $\mathrm{I}(M)$'s canonical iterations strategy.

If $\mathcal{T}$ does not have a fatal drop and some $Q \trianglelefteq \mathrm{I}(\mathcal{M}(\mathcal{T}))$ defines a counter example to $\delta(\mathcal{T})$ being a Woodin cardinal, then $\Sigma(\mathcal{T})$ is the unique branch b with $Q = Q(b, \mathcal{T})$.

If $\mathrm{I}(\mathcal{M}(\mathcal{T})) \models$ "$\delta(\mathcal{T})$ is Woodin", then let us write $\mathcal{Q} := \mathrm{I}^\omega(\mathcal{M}(\mathcal{T}))$. By strong iterability there exists for every $n < \omega$ branches that will move $\tau_j^{\mathcal{P}^\pi}$ to $\tau_j^{\mathcal{Q}}$ for all $j \leq n$.

Hence their restriction to $\mathrm{Hull}^{\mathcal{P}^\pi}(\gamma_n^{\mathcal{P}^\pi} \cup \{\tau_j^{\mathcal{P}^\pi} : j \leq n\})$ will be canonical. We can then string together these embeddings into

$$\sigma : \mathcal{P}^\pi = \mathrm{Hull}^{\mathcal{P}^\pi}(\delta^{\mathcal{P}^\pi} \cup \{\tau_j^{\mathcal{P}^\pi} : j < \omega\}) \to \mathrm{Hull}^{\mathcal{Q}}(\beta \cup \{\tau_j^{\mathcal{Q}} : j < \omega\})$$

where $\beta := \sup_{n<\omega}(\gamma_n^{\mathcal{Q}})$. We can do the same for the direct limit embedding from $\mathrm{Hull}^{\mathcal{Q}}(\gamma_n^{\mathcal{Q}} \cup \{\tau_j^{\mathcal{Q}} : j \leq n\})$ into $\mathcal{P}_\mu^X$ and thus get $\tau : \mathrm{Hull}^{\mathcal{Q}}(\beta \cup \{\tau_j^{\mathcal{Q}} : j < \omega\}) \to \mathcal{P}_\mu^X$ with $\pi = \tau \circ \sigma$. Let $\mathcal{Q}^*$ be the transitive collapse of $\mathrm{Hull}^{\mathcal{Q}}(\beta \cup \{\tau_j^{\mathcal{Q}} : j < \omega\})$ and let $\pi^* : \mathcal{Q}^* \to \mathcal{Q}$ be the reverse of the Mostowski collapse. Then the triangle of $(\pi^*)^{-1} \circ \sigma : \mathcal{P}^\pi \to \mathcal{Q}^*$, $\tau \circ \pi^* : \mathcal{Q}^* \to \mathcal{P}_\mu^X$ and $\pi : \mathcal{P}^\pi \to \mathcal{P}_\mu^X$ satisfies the requirements of Lemma 4.2, hence $\mathrm{I}(\mathcal{Q}^* || \beta) \subseteq \mathcal{Q}^*$ and by elementarity $\mathcal{Q}^* \models$ "β is Woodin". By suitabality of $\mathcal{Q}$ we must have $\mathcal{Q}^* = \mathcal{Q}$ and $\beta = \delta^{\mathcal{Q}}$. So we can amalgamate branches b_n with canonical images on γ^{P^π} into one b which is then unique as its image is cofinal in δ^M. Let this branch be $\Sigma(\mathcal{T})$.

Note that b can be defined from $\langle \dot{B}_i : i < \mu \rangle$ and does not depend on the choice of $\langle C_n : n < \omega \rangle$. It should also be easy to see that the $\dot{B}_i$ can be chosen to be sufficiently homogeneous s.t. the restriction of Σ to HOD_X is in HOD_X as desired. (We can pick names s.t.

$$\Vdash_{\mathrm{Col}(\omega,\mu)} \dot{B}_i = \{x \in \mathbb{R} | S_{\mu,\dot{g}}^X \models \varphi(x, \check{\beta}_i)\},$$

for β_i from M.) $\dashv$

Note that Σ^π has a realization property: for any $\mathcal{T}$ on P^π by Σ^π, if the tree embedding $\pi^{\mathcal{T}} : P^\pi \to \mathcal{M}^{\mathcal{T}}$ exists then there exist $\tau : \mathcal{M}^{\mathcal{T}} \to P_\mu^X$ s.t. $\pi = \tau \circ \pi^{\mathcal{T}}$.

Lemma 4.4: *Let Y be a good hull, $\pi : M \to Y$ be the reverse of the Mostowski collapse. Σ^π extends to a (κ, κ) iteration strategy $(\Sigma^\pi)^*$ which condenses for good hulls, i.e. if M is the transitive collapse of a good hull, then $((\Sigma^\pi)^*)^M = \Sigma^\pi \upharpoonright V_{\bar{\kappa}}^M$.*

Proof: The proof is by simultaneous induction: we can use the usual proof (see [Ste05] Lemma 2.15) for the extendibility, i.e. we find a good stable hull and amalgamate all the branches through hulls of our tree as long as we know that good hulls of our tree are actually according to Σ^π. So let us assume for some tree $\mathcal{T}^*$ we have just constructed a branch b^* using stable good hulls. Let us now take M the transitive collapse of a good hull containing $\mathcal{T}^*$ and b^*. Let us write $\mathcal{T}$ and b for the preimages.

Claim 1: $(\Sigma^\pi)^M = \Sigma^\pi \upharpoonright H_{\mu^+}^M$.

Proof of Claim: Because M is good we have that $\langle B_i \cap M[g] : i < \mu \rangle \in M[g]$ where $\langle B_i : i < \mu \rangle$ are sets that guide Σ^π as in the proof of Lemma 4.3, but from this it is easy for M to identify the correct branches. □

Let $\xi \in b$ be arbitrary. As b is cofinal it is enough to show that $\xi \in \Sigma^\pi(\mathcal{T})$. M will believe that there exists some stable good hull M' with preimage $\bar{\mathcal{T}}$ of $\mathcal{T}$ s.t. $\xi \in \pi_{M',M}$" $\left[(\Sigma^\pi)^M(\bar{\mathcal{T}})\right]$. But by elementarity M' is actually good and stable, so then, because $\mathcal{T}$ is a good hull above $\bar{\mathcal{T}}$, we'll have $\pi_{M',M}$" $\left[(\Sigma^\pi)^M(\bar{\mathcal{T}})\right] \subseteq \Sigma^\pi(\mathcal{T})$. $\dashv$

Let now Y be good and let $\pi : M \to Y$ be the reverse of the Mostowski collapse. Let $\langle \dot{B}_i : i < \mu \rangle$ be a sequence as in the proof of Lemma 4.3. By choice of the sequence, points defined from terms for the B_i are cofinal in $\delta^{\mathcal{P}^\pi}$.

By continuity at $\delta^{\mathcal{P}^\pi}$ these points are mapped cofinally into $\Theta^X_{\mu,g}$ by the direct limit map. We can thus fix a subsequence of which we can assume that for all good Y the appropriate sets of terms will contain that sequence. Henceforth this sequence shall be named $\langle \dot{B}_i : i < \mu \rangle$.

We will now define extended terms for $\langle \dot{B}_i : i < \mu \rangle$. They will be crucial in showing that Σ^π as above determines itself on generic extensions, though we will not be able to fully prove this until the end of this section.

Fix τ_i for $i < \mu$ a standard term capturing B_i. Let $\alpha < \kappa$, we define extended terms $\dot{B}_{i,\alpha}$ by $\mathbf{1} \Vdash_{\mathrm{Col}(\omega,\alpha)} \sigma \in \dot{B}_{i,\alpha}$ iff "there exists a non-dropping iteration tree $\mathcal{T}$ by $(\Sigma^\pi)^*$ of size $\leq \alpha$ s.t. $\mathbf{1} \Vdash_{\mathrm{Col}(\omega,\alpha)} \exists \dot{h} \subset \mathrm{Col}(\omega, i^{\mathcal{T}}(\check{\delta}^{P^X_0}))$ generic over $M^{\mathcal{T}}$ s.t. $\sigma^{\dot{G}} \in i^{\mathcal{T}}(\tau_i)^{\dot{h}}$".

Lemma 4.5: *$\mathcal{P}^X_\mu$ has a (κ,κ)-iteration strategy in* $\mathrm{HOD}_X[g]$.

PROOF: Let Y be good and let $\pi : M \to Y$ be the reversed Mostowski collapse. We will show that $\Sigma^\pi \restriction (V_{\bar{\kappa}})^{M[g]} \in M[g]$. By elementarity this will finish the proof.

Let $\mathcal{T}$ be by Σ^π in $M[g]$. We want to show that $\Sigma^\pi(\mathcal{T})$ is uniformly definable over $M[g]$. We want that $\mathcal{T}$ is countable there. If it is not fix $h \subseteq \mathrm{Col}(\omega, \mathcal{T})$ generic over $M[g]$ in $\mathrm{HOD}_X[g]$.

In $M[g][h]$ we can search for Q-structures in $\mathrm{I}(\mathcal{M}(\mathcal{T}))$. By Lemma 4.1 the Q-structure is in $M[g]$ and uniformly definable there. By absoluteness $\Sigma^\pi(\mathcal{T}) \in M[g][h]$ and by homeogeneity of the collapse then $\Sigma^\pi(\mathcal{T}) \in M[g]$.

If $\mathcal{T}$ is maximal consider the following: it should be easy to see that $(\dot{B}_{i,\alpha} \cap M)^h = B_i \cap M[g][h]$, this is because of the condensation given by Lemma 4.4; we can identify the target model of our branch easy enough, it is $N := \mathrm{I}^\omega(\mathcal{M}(\mathcal{T}))$ which ,crucially, is in $M[g]$; for every $i < \mu$ we have $\tau_i^N \in N$ a term for B_i; from the point of view of $M[g][h]$ we'll have "$(\tau_i^N)^{h'} = N[h'] \cap (\dot{B}_{i,\alpha} \cap M)^h$for all $h' \subset \mathrm{Col}(\omega, \delta^N)$ generic over N, i.e. τ_i^N is the standard term for $(\dot{B}_{i,\alpha} \cap M)^h$ from the point of view of $M[g][h]$. Thus $\Sigma(\mathcal{T})$ can be identified as the branch moving all those terms correctly. A standard absoluteness argument shows that it is in $M[g]$. ⊣

Our priority at this point will be to get an ω-suitable pair. We will first construct some ω-suitable premouse and some $(\omega_1, \omega_1 + 1)$ strategy for that premouse. Using the derived model theorem we can then show that some tail will satisfy branch condensation. From that point on it will be relatively simple to get generic extendability.

Take an increasing sequence $\langle \mu_n : n < \omega \rangle$ of good cardinals with $\mu_0 > \mu$. We will have $\sup\limits_{n<\omega} \mu_n < \kappa$. Fix some Θ-closed ordinal $\nu \in (\mu, \mu_0)$.

Inductively define $\mathcal{Q}(n+1)$ as $\mathcal{P}^X_{\mu_{n+1}}(\mathcal{Q}(n))$ $(\mathcal{Q}(-1) := \emptyset)$, let $\delta^{\mathcal{Q}}_n$ be the top Woodin cardinal of $\mathcal{Q}(n)$ $(\delta^{\mathcal{Q}}_{-1} := 0)$. Let $\mathcal{Q}^- := \bigcup\limits_{n<\omega} \mathcal{Q}(n)$ and let $\mathcal{Q}$ be the minimal segment $\mathcal{M}$ of $\mathrm{I}(\mathcal{Q}^-)$ s.t. $\rho_\omega(\mathcal{M}) < \sup\limits_{n<\omega} \delta^{\mathcal{Q}}_n$ if it exists, otherwise $\mathcal{Q} := \mathrm{I}(\mathcal{Q}^-)$. (Depending on one's precise definition of lower part closure like structures this may be redundant, but we want to be precise here.)

Let us now fix some good hull Y with $\pi : M \to Y$ the reverse of the Mostowski collapse and $\mathcal{Q} \in Y$. Let $\mathcal{P}$ be the preimage of $\mathcal{Q}$ under π. Let $\delta^{\mathcal{P}}_n$ refer to the Woodin cardinals of $\mathcal{P}$.

Lemma 4.6: *In any* $\mathrm{Col}(\omega, \nu)$*-generic extension of* $\mathrm{HOD}_X[g]$*,* $\mathcal{P}$ *has a fullness preserving iteration strategy defined for stacks of the form* $\langle \vec{\mathcal{T}}_\alpha : \alpha < \beta \rangle$ *where if* $\alpha + 1 < \beta$ *then* $\vec{\mathcal{T}}_\alpha$ *has a last model* $\mathcal{M}$*, the tree embedding* $\iota : \mathcal{P} \to \mathcal{M}$ *exists and* $\vec{\mathcal{T}}_{\alpha+1}$ *is a stack that concentrates on a window of the form* $\big(\iota(\delta_n^{\mathcal{P}}), \iota(\delta_{n+1}^{\mathcal{P}})\big)$ *for* $n \in \omega \cup \{-1\}$*. We will have that the restriction of this strategy to* $\mathrm{HOD}_X[g]$ *is in* $\mathrm{HOD}_X[g]$*.*

Proof: First note that by Lemma 4.5 applied to each $\mathcal{Q}(n)$ we have a (κ,κ)-iteration strategy Σ_n in $\mathrm{HOD}_X[g]$ for iteration trees based on the window $(\delta_n^{\mathcal{Q}}, \delta_{n+1}^{\mathcal{Q}})$ which extends to $\mathrm{Col}(\omega, \mu_n)$-generic extensions. We will only define the strategy for normal trees, it should be easy to see that we can extend this strategy to stacks of the required form as well.

Fix some $h \subseteq \mathrm{Col}(\omega, \nu)$ generic over $\mathrm{HOD}_X[g]$.

We will inductively define strategies for $\mathcal{P}(n)$ s.t. for every iteration based on $\mathcal{P}(n)$ with last model $\mathcal{R}$, $\mathcal{R}$ can be realized into $\mathcal{Q}(n)$. This not only guarantees well-foundedness (let σ be the iteration embedding, then $\mathrm{Ult}(\mathcal{P};\sigma)$ can be embedded into $\mathrm{Ult}(\mathcal{P};\pi)$.) but also fullness (by Lemma 4.2).

Now let us assume that the strategy for $\mathcal{P}(n)$ has already been defined. Let $\mathcal{T}$ be an iteration tree based on $\mathcal{P}(n+1)$ of size $<\nu^+$ in $\mathrm{HOD}_X[g][h]$, let $\mathcal{T}_n$ be the part of it that is based on $\mathcal{P}(n)$ with last model $\mathcal{R}$, iteration embedding σ and realization embedding τ. So we have the following commuting diagram:

$$\begin{tikzcd}
 & & \mathcal{Q}(n+1) \\
\mathcal{R} \arrow[urr, "\tau"] & & \\
 & \mathcal{P}(n+1) \arrow[ul, "\mathcal{T}_n", "\sigma"'] \arrow[uur, "\pi"'] &
\end{tikzcd}$$

Call the rest of the tree $\mathcal{T}^n$. We will want to iterate according to Σ_{n+1}^{τ}. Let us see that this works:

Take a good at μ_{n+1} hull of the whole situation. Let $\pi^* : M^* \to Y^*$ reverse the transitive collapse. $\mathcal{T} \in M^*[g][h]$. The above diagram then extends:

$$\begin{tikzcd}
 & & \mathcal{Q}(n+1) \\
 & & P^{\pi^*}(\mathcal{Q}(n)) \arrow[u, "\pi^*"'] \\
\mathcal{R} \arrow[urr, "(\pi^*)^{-1}\circ\tau"] & & \\
 & \mathcal{P}(n+1) \arrow[ul, "\mathcal{T}_n", "\sigma"'] \arrow[uur, "(\pi^*)^{-1}\circ\pi"'] &
\end{tikzcd}$$

Remember that $\mathcal{Q}(n+1) = \mathcal{P}^X_{\mu_{n+1}}(\mathcal{Q}(n))$. Now copy $\mathcal{T}^n$ onto $\mathcal{P}^{\pi^*}(\mathcal{Q}(n))$. We then have:

$$
\begin{tikzcd}
\mathcal{R}^{**} & \mathcal{P}^{\pi^*}(\mathcal{Q}(n)) \arrow[l, "(\mathcal{T}^n)^{(\pi^*)^{-1}\circ\tau}"', "\sigma^{**}"] \\
\mathcal{R}^* \arrow[u, "\tau^*"] & \mathcal{R} \arrow[l, "\mathcal{T}^n"', "\sigma^*"] \arrow[u, "(\pi^*)^{-1}\circ\tau"']
\end{tikzcd}
$$

If we assume that $\mathcal{T}^n$ is by Σ^{τ}_{n+1} then it should be easy to see that $(\mathcal{T}^n)^{(\pi^*)^{-1}\circ\tau}$ is by $\Sigma^{\pi^*}_{n+1}$ (this is because the cardinality of the tree is small).

But it is also by $(\pi^*)^{-1}(\Sigma_{n+1})$ - because of hull condensation relative to good hulls for Σ_{n+1} - which is a strategy which picks realizable branches. So we have:

$$
\begin{tikzcd}
 & & \mathcal{Q}(n+1) \\
\mathcal{R}^{**} \arrow[urr, "\tau^{**}"] & & \\
 & \mathcal{P}^{\pi^*}(\mathcal{Q}(n)) \arrow[ul, "(\mathcal{T}^n)^{(\pi^*)^{-1}\circ\tau}", "\sigma^{**}"'] \arrow[uur, "\pi^*"'] &
\end{tikzcd}
$$

Putting everything together we get:

$$
\begin{tikzcd}
 & & \mathcal{Q}(n+1) \\
 & \mathcal{R}^{**} \arrow[ur, "\tau^{**}"] & \\
 & & \mathcal{P}^{\pi^*}(\mathcal{Q}(n)) \arrow[ul, "(\mathcal{T}^n)^{(\pi^*)^{-1}\circ\tau}", "\sigma^{**}"'] \arrow[uu, "\pi^*"'] \\
\mathcal{R}^* \arrow[uur, "\tau^*"] & & \\
 & \mathcal{R} \arrow[ul, "\mathcal{T}^n", "\sigma^*"'] \arrow[uur, "(\pi^*)^{-1}\circ\tau"'] & \\
 & & \mathcal{P}(n+1) \arrow[ul, "\mathcal{T}_n", "\sigma"'] \arrow[uuu, "(\pi^*)^{-1}\circ\pi"']
\end{tikzcd}
$$

The outer triangle $\mathcal{P}(n+1), \mathcal{R}^*, \mathcal{Q}(n+1)$ with the maps $\sigma^* \circ \sigma, \tau^{**} \circ \tau^*, \pi$ is then as wanted. $\dashv$

Let us write Σ for the restriction of the above strategy to $\mathrm{HOD}_X[g]$.

We can now show that $\mathcal{Q}$ is not anomalous, i.e. $\mathcal{Q} = \mathrm{I}(\mathcal{Q}^-)$ or equivalently $\mathcal{P} = \mathrm{I}(\pi^{-1}(\mathcal{Q}^-)$. Otherwise let m be minimal s.t. $\rho_\omega(\mathcal{P}) < \delta_m^{\mathcal{P}}$, and let $\mathcal{P}^*$ be the appropriate core. We then have that $\mathcal{P}^*$ as a mouse over $\mathcal{P}(m)$ is of Lp-type. It then follows that by the previous lemma $\mathcal{P}^*$ is generically (ν, ν)-iterable in HOD_X. So by Lemma 3.4 we have a $(\mathrm{On}, \mathrm{On})$-strategy Λ for it. So then we'll have $\Lambda \restriction \mathbb{R} \in S^X_{\mu,g}$ by Lemma 3.8. The new set it defines, call it a, is then $OD^{S^X_{\mu,g}}(\mathcal{P}(m))$, on the other hand by mouse capturing we have $a \in \mathrm{I}(\mathcal{P}(m)) \subseteq \mathcal{P}$. Contradiction!

We are now going to borrow some notation from [Ste].

Lemma 4.7: *$D(\mathcal{P}, \lambda) \supseteq S^X_{\mu,g}$, where $\lambda := \sup_{n<\omega} \delta_n^{\mathcal{P}}$.*

Proof: Let M^* be the collapse of some good hull s.t. $\mathcal{P} \in M^*$. Set $\mathbb{R}^* = \mathbb{R} \cap M^*[g]$, and let $h \subset \mathrm{Col}(\omega, \mathbb{R}^*)$ be generic over $M^*[g]$. Let $\langle \mathcal{P}^n : n < \omega \rangle$ be a $\mathbb{R}^*$-genericity iteration in $M^*[g][h]$. By construction of Σ every $\mathcal{P}^n$ is realizable into $\mathcal{Q}$, and thus so is the direct limit $\mathcal{R}$ of this iteration.

By Lemma 4.2 we thus get $\mathcal{R} = \mathrm{I}^\omega(\mathcal{R}||\lambda^*)$ where λ^* is the image of λ under the direct limit embedding. Using S-constructions and mouse capturing we then get $\mathcal{R}(\mathbb{R}^*) \supseteq \mathrm{I}^\omega(\mathbb{R}^*)$, so if we can get $\mathrm{I}(\mathbb{R}^*) = (\mathrm{Lp}^+)^{M^*[g]}(\mathbb{R}^*)$ we are done. On the one hand surely $\mathrm{I}^{M^*}(\mathbb{R}^*) \trianglelefteq (\mathrm{Lp}^+)^{M^*[g]}(\mathbb{R}^*)$ but also $\mathrm{I}^{M^*}(\mathbb{R}^*) = \mathrm{I}(\mathbb{R}^*)$ by Lemma 4.1, and on the other hand $(\mathrm{Lp}^+)^{M^*[g]}(\mathbb{R}^*) \trianglelefteq \mathrm{I}(\mathbb{R}^*)$ by definiton of $(\mathrm{Lp}^+)^{\mathrm{HOD}_X[g]}(\mathbb{R})$. $\dashv$

Lemma 4.8: *Let $i < \mu$ and $n < \omega$. Then there exists a tail $(\mathcal{P}^*, \Sigma^*)$ of $(\mathcal{P}, \Sigma)$ s.t. for every tree $\mathcal{T}$ on $\mathcal{P}^*$ by Σ^* with a last model $\mathcal{Q}^*$ and branch embedding $\iota : \mathcal{P}^* \to \mathcal{Q}^*$, and every tree $\mathcal{U}$ on $\mathcal{P}^*$ of limit type and ι-realizable branch b, b does not drop and $i_b^{\mathcal{U}}$ moves $\tau_{B_i}^{\mathcal{P}^*(n)}$ correctly, i.e. to $\tau_{B_i}^{\mathcal{M}_b^{\mathcal{U}}(n)}$. Note that $\mathcal{M}_b^{\mathcal{U}}$ is ω-suitable by Lemma 4.2.*

Proof: This is Lemma 2.39 from [Sar14]. For the reader's convenience we will reproduce the argument here.

Assume not! Thus there exists a tuple $\langle \mathcal{P}^{0,k}, \mathcal{T}_k, \mathcal{U}_k, b_k, \sigma^{0,k} : k < \omega \rangle$ s.t.

- $\mathcal{P}^{0,0} = \mathcal{P}$;
- $\mathcal{P}^{0,k+1}$ is an iterate of $\mathcal{P}^{0,k}$, by that one's Σ-tail strategy as witnessed by $\mathcal{T}_k$, let $i^{0,k}$ be the iteration embedding;
- $\mathcal{U}_k$ is an iteration tree on $\mathcal{P}^{0,k}$ of limit type according to the Σ-tail strategy, b_k is a cofinal $i^{0,k}$-realizable branch through $\mathcal{U}_k$, $\sigma^{0,k} : \mathcal{R}^{0,k} := \mathcal{M}_{b_k}^{\mathcal{U}_k} \to \mathcal{P}^{0,k+1}$ is the realization embedding;
- $j^{0,k}(:= i_{b_k}^{\mathcal{U}_k})(\tau_{B_i}^{\mathcal{P}^{0,k}(n)}) \neq \tau_{B_i}^{\mathcal{R}^{0,k}(n)}$.

Let $\mathcal{P}^{0,\omega}$ be the direct limit. We will inductively define a genericity iterations $\langle \mathcal{P}^{l,k} : l < \omega \rangle$ and $\langle \mathcal{R}^{l,k} : l < \omega \rangle$ on $\mathcal{P}^{0,k}$ above $\mathcal{P}^{0,k}(n)$ and $\mathcal{R}^{0,k}$ above $\mathcal{R}^{0,k}(n)$ respectively. Assume that $\langle \mathcal{P}^{l,k} : k < \omega \rangle$ is already defined. First we will copy it onto $\mathcal{R}^{0,k}$ using $j^{0,k}$. This works because the Σ-tail strategies are pullbacks of a strategy on $\mathcal{Q}$ and $\mathcal{R}^{0,k}$ is iterated by the $\sigma^{0,k}$-pullback of $\mathcal{P}^{0,k+1}$'s iteration strategy. Crucially, all the maps commute. This copy-iteration is immediately followed by a standard genericity iteration.

Let us write $\mathcal{P}^{\omega,k}$ for the direct limit of the $\langle \mathcal{P}^{l,k} : l < \omega \rangle$ and $j^{\omega,k} : \mathcal{P}^{\omega,k} \to \mathcal{R}^{\omega,k}$ and $\sigma^{\omega,k} : \mathcal{R}^{\omega,k} \to \mathcal{P}^{\omega,k+1}$ for the copy maps. Let us write $\mathcal{P}^{\omega,\omega}$ for the direct limit of the $\mathcal{P}^{\omega,k}$ under $i^{\omega,k}$.

Claim 1: *$\mathcal{P}^{\omega,\omega}$ is well-founded.*

Proof of Claim: Let $\langle \mathcal{P}^{l,\omega} : l < \omega \rangle$ be the result of copying all the $\langle \mathcal{P}^{l,k} : l < \omega \rangle$ onto $\mathcal{P}^{0,\omega}$ in sequence. We will then have that $\mathcal{P}^{\omega,\omega}$ is isomorphic to the direct limit of $\langle \mathcal{P}^{l,\omega} : l < \omega \rangle$. By the above this is essentially an iteration by Σ and thus well-founded.$\square$

$$\begin{tikzcd}
\mathcal{P}^{\omega,0} \arrow[r, "j^{\omega,0}"] \arrow[d, dotted, no head] & \mathcal{R}^{\omega,0} \arrow[r, "\sigma^{\omega,0}"] \arrow[d, dotted, no head] & \mathcal{P}^{\omega,1} \arrow[r, dotted, no head] \arrow[d, dotted, no head] & \mathcal{P}^{\omega,\omega} \arrow[d, dotted, no head] \\
\mathcal{P}^{1,0} \arrow[r, "j^{1,0}"] & \mathcal{R}^{1,0} \arrow[r, "\sigma^{1,0}"] & \mathcal{P}^{1,1} \arrow[r, dotted, no head] & \mathcal{P}^{1,\omega} \\
\mathcal{P}^{0,0} \arrow[r, "j^{0,0}"] \arrow[u] & \mathcal{R}^{0,0} \arrow[r, "\sigma^{0,0}"] \arrow[u] & \mathcal{P}^{0,1} \arrow[r, dotted, no head] \arrow[u] & \mathcal{P}^{0,\omega} \arrow[u]
\end{tikzcd}$$

Let us now fix some $\xi < \Theta^X_\mu$ s.t.

$$B_i = \{x \in \mathbb{R} |\, \mathrm{Lp}(\mathbb{R}) \models \varphi(x,\xi)\}.$$

By the claim there must exist some $k < \omega$ s.t. $i^{\omega,k}$ fixes ξ. On the other hand $\mathcal{P}^{\omega,k}$ believes: " $\tau^{\mathcal{P}^{\omega,k}(n)}_{B_i}$ is the term for the set of reals satisfying $\varphi(\cdot,\xi)$ in $S^X_{\mu,g}$", and so does $\mathcal{P}^{\omega,k+1}$. Note that by the previous lemma $S^X_{\mu,g}$ is uniformly definable in the derived model of $\mathcal{P}^{\omega,k}$. Thus $j^{\omega,k}(\tau^{\mathcal{P}^{\omega,k}(n)}_{B_i}) = \tau^{\mathcal{R}^{\omega,k}(n)}_{B_i}$, but then agreement between $j^{\omega,k}$ and $j^{0,k}$ gives a contradiction! $\dashv$

Corollary 4.9: *Some tail $(\mathcal{P}^*, \Sigma^*)$ of (P, Σ) has branch condensation.*

Proof: By chaining iterations we can find a tail $(\mathcal{P}^*, \Sigma^*)$ that satisfies the lemma for all B_i simultaneously. We will see that this works. First note that by letting $\mathcal{U}$ be $\mathcal{T}$ without its last branch, $b = \Sigma^*(\mathcal{U})$ and σ the identity we see that Σ^* picks branches that move all terms correctly (unless there is a drop). Because terms define a cofinal subset of our Woodin cardinals, this completley determines Σ^*.

Now, if $(\mathcal{P}^{**}, \Sigma^{**})$ is a tail of $(\mathcal{P}^*, \Sigma^*)$ and $\mathcal{U}$ is a tree on $\mathcal{P}^*$ by Σ^* and b is a branch through $\mathcal{U}$ that can be realized into $\mathcal{P}^{**}$ then b moves all terms correctly and hence $b = \Sigma^*(\mathcal{U})$. $\dashv$

W.l.o.g. assume that $\mathcal{P}^* = \mathcal{P}$ and $\Sigma^* = \Sigma$. We can now almost finish the proof.

Lemma 4.10: *Σ determines itself on generic extensions.*

Proof: Let Y^* be some good hull and $\pi^* : M^* \to Y^*$ the reverse of the Mostowski collapse. Let $h \in \mathrm{HOD}_X[g]$ be generic over $M^*[g]$ for some $<\bar{\kappa}$ forcing notion.

Looking at the proof of Lemma 4.5 we can easily see that $\Sigma \restriction M^* [g] \in M^* [g]$, furthermore the only obstacle to finishing the proof is that we lack a reliable way to identify $\mathrm{I}(a)$ for $a \in M^* [g] [h]$.

We have that $\langle B_i \cap M^* [g] [h] : i < \mu \rangle \in M^* [g] [h]$ using $\langle \dot{B}_{i,\alpha} : i < \mu, \alpha < \kappa \rangle \in Y^*$. Using genericity iterations we can show that

$$(H_{\omega_1}^{M^*[g]}; \in, (B_i \cap M^* [g] : i < \mu)) \prec (H_{\omega_1}^{M^*[g][h]}; \in, (B_i \cap M^* [g] [h] : i < \mu)).$$

We only need to have a strategy in $M^* [g]$ as we can always make names for reals in $M^* [g] [h]$ generic. Standard facts about capturing sets at Woodin cardinals then do the rest. (Note: For example see [SS] Section 1.4)

Now w.l.o.g. we can assume that

$$B_0 = \{(x, y) : x \text{ codes } a \in H_{\omega_1}, y \text{ codes} \operatorname{Lp}^{\Gamma_\mu^X}(a)\}.$$

By the above we then have for all $a \in H_{\omega_1}^{M^*[g][h]}$ that for all reals x coding a there exists some real y s.t. $(x, y) \in B_0 \cap M^* [g] [h]$. Any such y will then code $\operatorname{Lp}^{\Gamma_{\mu,g}^X}(a) = \mathrm{I}(a)$ as needed. $\dashv$

Σ then extends to a $(\mathrm{On}, \mathrm{On})$-iteration strategy over $V[g]$ and by Lemma 3.5 we have $M_\omega^{\Lambda,\#}$. Furthermore, $\Sigma \restriction \mathbb{R}$ can not be in $S_{\mu,g}^X$ because it defines a prewellorder of length Θ_μ^X. Thus,

$$L(\Sigma \restriction \mathbb{R}, \mathbb{R}) \models \mathrm{AD}^+ + \Theta > \theta_0.$$

5 Reaching the limit stage

Let now $\alpha < \kappa$ we want to show that whenever $h \subset \mathrm{Col}(\omega, \alpha)$ is generic over V we have that Σ extends to a ZFC-fullness preserving (κ, κ)-iteration strategy. W.l.o.g. α is good.

Let Σ^h be the extension of Σ to $\mathrm{HOD}_X [h]$. Let $M := (\mathrm{Lp}^+)^{\mathrm{HOD}_X[]}(\mathbb{R}^{\mathrm{HOD}_X[h]})$. It will be enough to show that Σ^h is $\Gamma_{\alpha,h}^X$-fullness preserving by the results of Section 2 and the fact that good α's are unbounded in κ.

Let D^* be the derived model over $(\mathcal{P}, \Sigma^h)$ in $\mathrm{HOD}_X [h]$. Remembering our extended terms $\langle \dot{B}_{i,\alpha} : i < \mu \rangle$ let $B_i^* := \dot{B}_{i,\alpha}^h$ and Γ^* be the pointclass in $\mathrm{HOD}_X [h]$ generated by the B_i^*. We'll have $\Gamma^* \subset D^*$ because Σ^h does move terms for B_i^* correctly.

On the other hand we'll have $L(B_i^*, \mathbb{R}^{\mathrm{HOD}_X[h]}) \models$ " $\mathrm{AD}^+ + \Theta = \theta_0$" for all $i < \mu$, hence $\Gamma^* \subseteq \Gamma_{\alpha,h}^X$. As D^* believes "$\mathcal{P}$ is strongly B_i^*-iterable" for all $i < \mu$, it will be enough to show that equality holds.

Assume not. Then there is a set $A \in \Gamma_{\alpha,h}^X$ that every set in Γ^* is Wadge reducible to. But Σ^h on countable trees can be computed from Γ^*, hence $\Sigma^h \restriction H_{\omega_1}^{\mathrm{HOD}_X[h]} \in M$.

Let now be $\mathcal{Q}$ be an element of HOD-directed limit of M. Let Y be a good at α hull containing a name for $\mathcal{Q}$. Let $\mathcal{Q}^*$ be the image of $\mathcal{P}_{\alpha,h}^X$ under the transitive collapse. By the results of the previous section we have that $\mathcal{Q}^*$ has a fullness preserving iteration strategy. Also, $\mathcal{Q}^*$ is clearly a pseudo-iterate of $\mathcal{Q}$.

We now want to compare $\mathcal{Q}^*$ with $\mathcal{P}(0)$. We want to show that they iterate to a commom model. Note that the co-iteration takes place in HOD_X. Let $\mathcal{T} \in \mathrm{HOD}_X$ be a

$<\kappa$-iteration tree on $\mathcal{P}$ by Σ. Take a good at μ hull Y containing $\mathcal{T}$. Let $\pi : N \to Y$ be the reverse of the Mostowski collapse. Then $\pi^{-1}(\mathcal{T})$ will be by Σ and thus its last model (if it exists) will be full. By Lemma 4.1 N recognizes that fact and reflects it upward.

In conclusion, any $<\kappa$-iterate of $\mathcal{P}$ by Σ in HOD_X (!) is (ZFC)-full. By standard arguments then Q^* and $\mathcal{P}(0)$ iterate to a common model which is a pseudo-iterate of $\mathcal{Q}$. Notice then that the direct limit of countable iterates of $\mathcal{P}$ by Σ^h in $\mathrm{HOD}_X[h]$ computes a pre-wellorder of length at least $\Theta^X_{\alpha,h}$. But this direct limit can be computed in M. Contradiction!

We could now try to repeat the argument relative to some $(\mathcal{P}, \Sigma)$ as above to get a model of $\mathrm{AD}^+ + \Theta > \theta_1$. To do that we now have to pick X' that is cofinal in $\mathrm{Lp}^{\Sigma}(A')$ where A' codes $V_\kappa^{\mathrm{HOD}_X}$ in some straightforward fashion. As we do not have even minimal amounts of choice in our ground model this approach will never get us beyond finite stages of this process. We have no choice but to isolate our HOD-pair from the choice of X.

Lemma 5.1: *Let $X \subset \mathrm{On}$ and let $\eta < \kappa$. Let $\mathcal{P} \in H^{\mathrm{HOD}_X}_{\eta^+}$ and Σ a (η^+, η^+)-strategy over HOD_X be s.t.*

- *$(\mathcal{P}, \Sigma)$ is a HOD-pair, λ^P is limit of non-measurable cofinality in P, Σ has branch condesation, determines itself on generic extension and is ZFC-fullness preserving for $<\kappa$-iterates absolute to $<\kappa$-generic extensions, i.e. whenever $\mathcal{T}$ is a tree on P by Σ of length $<\kappa$ with last model Q the main branch does not drop, then $\mathrm{I}^{\Sigma_{Q(\alpha)^-}}(Q||\gamma) \subseteq Q$ for all $\beta \leq \lambda^Q$ and all cutpoints γ above $Q(\beta)^-$,*
- *or $(\mathcal{P}, \Sigma)$ is a ω-suitable Λ-pair s.t. Λ has branch condensation, determines itself on generic extensions and is fullness preserving for $<\kappa$-iterates absolute to $<\kappa$-generic extensions where (Q, Λ) is a HOD-pair s.t. Λ has branch condensation, determines itself on generic extension and extends to an OD over V $(\mathrm{On}, \mathrm{On})$-iteration strategy.*

Then there exists a tail $(\mathcal{P}^*, \Sigma^*)$ which is OD over V.

Proof: Let $g \subset \mathrm{Col}(\omega, \eta)$ be generic over V. We define a pointclass Γ: in both cases we can form a derived model over (P, Σ) in $\mathrm{HOD}_X[g]$, call it D. Let $\Gamma := \mathcal{P}(\mathbb{R}) \cap D$.

Let $A \in \Gamma$, then A is universally Baire. Let α be an ordinal write Q_α for the unique Q s.t.

$$\mathbf{1} \Vdash_{\mathrm{Col}(\omega,\alpha)} \check{Q} \text{ is the direct limit of all } <\alpha - \Sigma^{\dot{G}}\text{-iterates}$$

holds over $\mathrm{HOD}_X[g]$. Write then T^A_α for the tree searching for $x \in \mathbb{R}$, a countable nowhere dropping iteration tree $\mathcal{T}$ on P with last branch b, i_{P,Q_α}-realisation embeddings for every limit stage of $\mathcal{T}$ including for b (certifying that $\mathcal{T}$ is by Σ) and some generic h over $M_b^{\mathcal{T}}$ with $x \in (i_b^{\mathcal{T}}(\tau_A^P))^h$. A complementing tree U^A_α is defined similarly.

Because of the Vopenka algebra this u.B. presentation also represents a u.B. set over $V[g]$, call this extension A^*. Building the derived model of P over V will show that $L(A^*, \mathbb{R} \cap V[g]) \models \mathrm{AD}^+$. We can assume that there are no diverging models of AD, so we have Γ^* extension of Γ, all sets u.B., determined and well-foundededly Wadge comparable.

Note now that the sequence of Q_α for Θ-closed α is OD. That is because if (P^*, Σ^*) were another pair like it at the same μ but possibly over some HOD_Y generating the same pointclass, they can be compared in V. The successful co-iteration would then be $<\alpha$-generic because of the Vopenka algebra. Crucially, here Γ^* is definable from its Wadge rank.

The same holds true for the sequence $\pi_{\alpha,\beta} : Q_\alpha \to Q_\beta$, $\alpha < \beta$ Θ-closed, of the iteration embeddings inbetween them.

The tail Σ_α on Q_α for $<\kappa$ trees can then be defined thusly: let $\beta > \mathrm{lh}(\mathcal{T})$, $\mathcal{T}$ by Σ_α, $\Sigma_\alpha(\mathcal{T})$ is then the unique branch b s.t. there exists $\sigma : \mathcal{M}_b^{\mathcal{T}} \to Q_\beta$ with $\pi_{\alpha,\beta} = \sigma \circ i_b^{\mathcal{T}}$ or b has a generically $(\mathrm{On}, \mathrm{On})$-iterable Q-structure. $\dashv$

Lemma 5.2: *Let $\mathcal{P} \in H_{\eta^+}^{\mathrm{HOD}}, \Sigma$ be as above. Let $A \subset \kappa$ code V_κ^{HOD} in some straightforward fashion, let $X \subset \mathrm{Lp}^{\Sigma}(A)$ be cofinal of ordertype ω. Let $\mu < \kappa$ be $\max(\zeta, \eta)$-closed in HOD_X. Then there exists (Q, Λ) a ω-suitable Σ-pair s.t. Λ has branch condensation, determines itself on generic extensions and is* ZFC*-fullness preserving for $<\kappa$-trees absolute to $<\kappa$-generic extensions.*

PROOF: This is just a rehash of the previous two and a half sections. Let $g \subset \mathrm{Col}(\omega, \mu)$ be generic over V. We work with $M := (\mathrm{Lp}^{+,\Sigma})^{\mathrm{HOD}_X[g]}(\mathbb{R}^{\mathrm{HOD}_X[g]}, \Sigma \restriction H_{\omega_1}^{\mathrm{HOD}_X[g]})$. This works the same as $(\mathrm{Lp}^+)^{\mathrm{HOD}_X[g]}(\mathbb{R})$ in the previous sections, using the appropriate results from [STb].

Note that the tree of the scale on a universal $(\Sigma_1^2(\Sigma))^M$ set only exists in $\mathrm{HOD}_X\,[g \restriction \eta]$. So we need to make sure that our good hulls are still countably closed in $\mathrm{HOD}_X\,[g \restriction \eta]$. We skip further details. $\dashv$

Lemma 5.3: *Let $(\mathcal{P}, \Sigma) \in H_{\eta^+}^{\mathrm{HOD}}$ be as above. Then there exists a $\Sigma-\mathrm{HOD}$-pair $(\mathcal{Q}, \Lambda) \in V_\kappa^{\mathrm{HOD}}$ s.t. $\lambda^{\mathcal{Q}} = \omega$, Λ has branch condensation, determines itself on generic extensions and is* ZFC*-fullness preserving for $<\kappa$-trees absolute to $<\kappa$-generic extension.*

PROOF: Working in HOD, by the above lemmata we can find a sequence $\langle \eta_i : i < \omega \rangle$ and names $\langle \dot\Gamma_i : i < \omega \rangle$ s.t.

- $\dot\Gamma_i$ is a determined u.B. pointclass in $\mathrm{HOD}^{\mathrm{Col}(\omega,\eta_i)}$ for all $i < \omega$, let $\dot\Gamma_i^\alpha$ be the extension to $\mathrm{HOD}^{\mathrm{Col}(\omega,\alpha)}$ for $\alpha > \eta_i$;
- $\mathrm{Lp}^{\dot\Gamma_i^\alpha,\Sigma}(a) = \mathrm{I}^{\dot\Sigma}(a)$ for all $a \in H_{\omega_1}$ and $\Sigma \in \dot\Gamma_i$, in $\mathrm{HOD}^{\mathrm{Col}(\omega,\alpha)}$ for all $\alpha < \kappa$ and $i < \omega$;
- $\mathrm{Lp}^{\dot\Gamma_i^{\eta_{i+1}},\Sigma}(a) = \mathrm{Lp}^{\dot\Gamma_{i+1},\Sigma}(a)$ for all $a \in H_{\omega_1}$ and $\Sigma \in \dot\Gamma_i$, in $\mathrm{HOD}^{\mathrm{Col}(\omega,\eta_{i+1})}$ for all $i < \omega$;
- Wadge degree of $\dot\Gamma_i^{\eta_{i+1}}$ less than Wadge degree of $\dot\Gamma_{i+1}$ in $\mathrm{HOD}^{\mathrm{Col}(\omega,\eta_{i+1})}$.

We can assume that $\eta^+ := \sup_{i<\omega} \eta_i < \kappa$. Let $h \subseteq \mathrm{Col}(\omega, \eta^+)$ be generic. Let $(\mathcal{Q}, \bigoplus_{\alpha<\lambda^{\mathcal{Q}}} \Lambda_\alpha)$ be the direct limit under co-iteration of all Σ-HOD-pairs $(Q, \Lambda) \in \mathrm{HP}^{\mathrm{HOD}[h \restriction \eta_i]}(\omega_1)$ s.t. Λ has branch condensation, determines itself on generic extensions, and is fullness preserving for $<\kappa$-trees absolute to $<\kappa$-generic extensions for some i.

We can assume that $\mathcal{Q}$ has exactly ω Woodin cardinals above $\mathcal{P}$, otherwise there is nothing left to show. Notice that $\mathcal{Q} \in \mathrm{HOD}$ and so are the restrictions of Λ_α to HOD. Write $\Lambda := \bigoplus_{n<\omega} \Lambda_n$. Let now A code V_κ^{HOD} in some straightforward fashion. Let

$X \subseteq \mathrm{Lp}^{\Lambda}(A)$ be cofinal of ordertype ω. Let $\eta^+ < \mu < \kappa$ be some Θ-closed cardinal that is ζ-closed in HOD_X. Let $g \subset \mathrm{Col}(\omega, \mu)$ be generic over V.

Working in $\mathrm{HOD}_X[g]$, let $(\mathcal{Q}^+, \Lambda^+)$ be the direct limit of all countable Σ-iterates. We say some hull Y is good iff it is closed under $\max\{\zeta, \eta^+\}$-sequences, $\mathcal{Q}^+, \mathrm{Lp}^{\Lambda}(A) \in Y$, $\mu \subset Y$ and Y has size μ. We will show that some preimage of Q^+ under a good hull is as wanted.

Let $\mathcal{Q}^{++} := \mathrm{I}^{\Lambda^+}(\mathcal{Q}^+)$. Let Y be a good hull, and let $\pi : M \to Y$ be the reverse of the Mostowski collapse. Let $(\mathcal{Q}^\pi, \Lambda^\pi)$ be the π-pullback of $(\mathcal{Q}^{++}, \Lambda^+)$. Similar to the proof of Lemma 4.1 we can show that M is closed under I^{Λ}, but this also gives closure under I^{Λ^*} whenever $(\mathcal{Q}^*, \Lambda^*)$ is a tail of $(\mathcal{Q}, \Lambda)$ in M, as Λ^* is OD in Λ and $\mathcal{Q}^*$. (A result of positionality for HOD-pairs, see Lemma 2.39.)

We now claim that Λ^π is a fullness preserving iteration strategy on $\mathcal{Q}^{++}$. There is a canonical candidate for an iteration strategy. We only have to show that given $\mathcal{T}$ on $\mathcal{Q}^\pi(n)$ by Λ^π_n with non-dropping last branch b, we have $\mathrm{Ult}(\mathcal{Q}^\pi, E_{i^{\mathcal{T}}_b})$ is wellfounded and full. But $\mathcal{Q}^\pi(n)$ is a tail of $(\mathcal{Q}(n), \Lambda_n)$, so b is π-realizable, and hence that ultrapower is realizable into $\mathcal{Q}^{++}$. Iterability then follows easily.

Note that we easily get that Λ^π is fullness preserving on $\mathcal{Q}^\pi(n)$ for any $n < \omega$. To show fullness preservation at the top is more tricky.

First let us show that $\rho_\omega(\mathcal{Q}^\pi) \geq \delta^{\mathcal{Q}^\pi}_\omega$. Assume not. Let n be minimal s.t. $\rho_\omega \leq \delta^{\mathcal{Q}^\pi}_n$. Let $(\mathcal{R}, \Phi)$ be the appropriate core, and a the new set that is defined over $\mathcal{R}$. We have that $M^{\Phi,\#}_\omega$ exists and hence $L(\mathbb{R}^{\mathrm{HOD}_X[g]}, \Phi \restriction H^{\mathrm{HOD}_X[g]}_{\omega_1}) \models \mathrm{AD}$. By maximality of $\Gamma := \dot{\Gamma}^\mu_{n+1}$ we'll have $\Phi \in \Gamma$ and hence that a is $\mathrm{OD}^{L(\Gamma,\mathbb{R})}_{\Lambda_n}$. On the other hand iterating $\mathcal{Q}^\pi$ above $\delta^{\mathcal{Q}^\pi}_n$ will generate $\mathrm{HOD}^{L(\Gamma,\mathbb{R})}_{\Lambda_n}$ which means that a is still a "new" set over it. Contradiction!

We'd like to use Lemma 4.2 to prove fullness at the top, but we need to make some adjustment to the proof: We'd like a tree that projects to the set of quadruplets (x, y, z, q) s.t. r codes an HOD initial segment of $(\mathcal{R}, \Lambda^*)$, some Λ-tail, x codes some set in H_{ω_1}, $y, z \trianglelefteq \mathrm{Lp}^{\Lambda^*}(x)$ and $y \trianglelefteq z$. This set is OD_Λ. This uses that strategies from HOD-pairs are positional, otherwise we would have to put in the iteration going from $\mathcal{Q}$ to $\mathcal{R}$ as well.

Let $N := (\mathrm{Lp}^{+,\Lambda})^{\mathrm{HOD}_X[g]}(\mathbb{R}^{\mathrm{HOD}_X[g]}, \Lambda \restriction H^{\mathrm{HOD}_X[g]}_{\omega_1})$. We can use a tree T on the universal $(\Sigma^2_1(\Lambda, \cdot))^N$ set which exists in $\mathrm{HOD}_X[g \restriction \eta^+]$. Crucially, $M[g \restriction \eta^+]$ is still ω-closed.

Clearly, Λ^π determines itself on generic extensions, as its components do. We'll skip further details. $\dashv$

6 Up to "Θ regular"

Now, let $(\mathcal{Q}^X_\alpha, \Lambda^X_\beta)$ be the unique $(\mathcal{Q}, \Lambda)$ s.t. in any $\mathrm{Col}(\omega, \alpha)$-generic extension of HOD_X $(\mathcal{Q}, \Lambda)$ is the HOD-limit of all HOD-pairs $(\mathcal{P}, \Sigma)$ s.t. Σ has branch condensation, determines itself on generic extensions and is fullness preserving for $<\kappa$-trees absolute to $<\kappa$-generic extensions. Here $\alpha < \kappa$ is Θ-closed and $X \subset \mathrm{On}$, if $X = \emptyset$ we will drop it from the notation. If β is a limit of Θ-closed cardinals, we let $(\mathcal{Q}^X_{<\beta}, \Lambda^X_{<\beta})$ be the obvious thing. We'll have $(\mathcal{Q}^X_\alpha, \Lambda^X_\alpha \restriction \mathrm{HOD}_X) \in \mathrm{HOD}_X$.

Let $\lambda^X_\alpha := \lambda^{\mathcal{Q}^X_\alpha}$. Whenever $\alpha < \beta$ are Θ-closed we have an iteration embedding $\sigma^X_{\alpha,\beta} : \mathcal{Q}^X_\alpha \to \mathcal{Q}_\beta$ that is OD_X by the results of the previous sections.

Let now $A \subset \kappa$ code V^{HOD}_κ in some straightforward fashion and let $X \subset \mathrm{Lp}^{\Lambda_{<\kappa}}(A \cup \{\mathcal{Q}_{<\kappa}\})$ be cofinal of ordertype ω. Let $\mu < \kappa$ be ζ-closed in HOD_X. For convenience's sake we will drop the subscripts in $\mathcal{Q}_{<\kappa}$ and $\Lambda_{<\kappa}$.

We say a $Y \prec H^{\mathrm{HOD}_X}_\eta$ for some carefully chosen η -note again that there are club many such η- is good iff Y is ζ-closed, $\mathrm{Lp}^\Lambda(\mathcal{Q}), \mathrm{Lp}^\Lambda(A) \in Y$, $\mu \subset Y$ and Y has size μ. As usual, we will write $\bar{x}$ for the image of any $x \in Y$ under the transitive collapse.

Lemma 6.1: *Let $\alpha < \lambda$, then $\mathcal{Q}||((\delta^{\mathcal{Q}}_\alpha)^+)^{\mathcal{Q}}$ is the stack of all $\Lambda_{\mathcal{Q}(\alpha)^-}$-premice $\mathcal{M}$ s.t. $\bar{\mathcal{M}} \trianglelefteq \mathrm{I}^{\Lambda^\tau_{\mathcal{Q}(\alpha)^-}}(\tau^{-1}(\mathcal{Q}||\delta^{\mathcal{Q}}_\alpha))$ for all $\tau : \bar{\mathcal{M}} \to \mathcal{M}$ of size β where β is minimal s.t. $\mathcal{Q}(\alpha)$ has a preimage of size β in HOD_X.*

Proof: Let us fix $\alpha < \lambda$ and $\beta < \kappa$ as above. Let us first consider some $\mathcal{M} \trianglelefteq \mathcal{Q}||((\delta^{\mathcal{Q}}_\alpha)^+)^{\mathcal{Q}}$, and let $\tau : \bar{\mathcal{M}} \to \mathcal{M}$ be a hull of size β in HOD_X.

Let $Y \prec H^{\mathrm{HOD}_X}_\eta$ be of size β containing both τ and some preimage $(\mathcal{P}, \Sigma)$ of $(\mathcal{Q}(\alpha), \Lambda_\alpha)$ where $\mathcal{P}$ has size β, Σ has branch condensation, determines itself on generic extensions and is fullness preserving for $<\kappa$-trees absolute to $<\kappa$-generic extensions.

Let $\pi : M \to Y$ be the reverse of the Mostowski collapse. We'll have that $\pi^{-1}(\mathcal{Q}(\alpha))$ is an actual Σ-iterate of $\mathcal{P}$. By fullness preservation of Σ we get that $\pi^{-1}(\mathcal{M}) \trianglelefteq \mathrm{I}^{\Lambda^\pi_{\mathcal{Q}(\alpha)^-}}(\pi^{-1}(\mathcal{Q}||\delta^{\mathcal{Q}}_\alpha))$. $\bar{\mathcal{M}}$ is then a hull of $\pi^{-1}(\mathcal{M})$ as witnessed by $\pi^{-1}(\tau)$ and hence $\bar{\mathcal{M}} \trianglelefteq \mathrm{I}^{(\Lambda^\pi_{\mathcal{Q}(\alpha)^-})^{\pi^{-1}(\tau)}}((\pi^{-1}(\tau))^{-1}(\pi^{-1}(\mathcal{Q}||\delta^{\mathcal{Q}}_\alpha)))$ and therefore $\bar{\mathcal{M}} \trianglelefteq \mathrm{I}^{\Lambda^\tau_{\mathcal{Q}(\alpha)^-}}(\tau^{-1}(\mathcal{Q}||\delta^{\mathcal{Q}}_\alpha))$ as wanted.

Now let $\mathcal{M}$ be with the above property. We will show that $\mathcal{M} \trianglelefteq \mathcal{Q}||((\delta^{\mathcal{Q}}_\alpha)^+)^{\mathcal{Q}}$. Let Y be a hull as above with $\mathcal{M} \in Y$ and $\pi : M \to Y$ be the reverse of the Mostowski collapse. We have that $\pi^{-1}(\mathcal{M}) \trianglelefteq \mathrm{I}^{\Lambda^\pi_{\mathcal{Q}(\alpha)^-}}(\pi^{-1}(\mathcal{Q}||\delta^{\mathcal{Q}}_\alpha))$. On the other hand $\pi^{-1}(\mathcal{Q}(\alpha))$ is a Σ-iterate and hence $\pi^{-1}(\mathcal{Q}||((\delta^{\mathcal{Q}}_\alpha)^+)^{\mathcal{Q}}) = \mathrm{I}^{\Lambda^\pi_{\mathcal{Q}(\alpha)^-}}(\pi^{-1}(\mathcal{Q}||\delta^{\mathcal{Q}}_\alpha))$. So, $\pi^{-1}(\mathcal{M}) \trianglelefteq \pi^{-1}(\mathcal{Q}||((\delta^{\mathcal{Q}}_\alpha)^+)^{\mathcal{Q}})$ and hence $\mathcal{M} \trianglelefteq \mathcal{Q}||((\delta^{\mathcal{Q}}_\alpha)^+)^{\mathcal{Q}}$. ⊣

Lemma 6.2: *Let Y be a good hull, $\pi : M \to Y$ be the reverse of the Mostowski collapse. Let $\alpha < \bar{\lambda}$. Let $\mathcal{T}$ be an iteration tree on $\mathcal{Q}^\pi(\alpha) := \pi^{-1}(\mathcal{Q}(\pi(\alpha)))$ by $\Lambda^\pi_{\pi(\alpha)}$ of length $<\kappa$ existing in some $<\kappa$-generic extension s.t. $\mathcal{T}$ has a last model and there is no drop on the main branch. Then there exists some $\sigma : \mathcal{M}^{\mathcal{T}} \to \mathcal{Q}(\pi(\alpha))$ s.t. $\pi = \sigma \circ i^{\mathcal{T}}$, where $\mathcal{M}^{\mathcal{T}}$ is the last model and $i^{\mathcal{T}}$ the branch embedding. So, Λ^π is a π-realization strategy.*

Proof: Let $\beta < \kappa$ be a Θ-closed cardinal s.t. $\mathcal{Q}(\pi(\alpha))$ is the tail of some $(\mathcal{P}, \Sigma) \in \mathrm{HP}^{\mathrm{HOD}_X}(\beta^+)$ s.t. Σ has branch condensation, determines itself on generic extensions and is fullness preserving for $<\kappa$-trees absolute to $<\kappa$-generic extensions. We can also assume that $\mathcal{T}$ exists in $\mathrm{HOD}_X[h]$ where $h \subset \mathrm{Col}(\omega, \beta)$ is generic over V.

We start with the following situation working in $\mathrm{HOD}_X[h]$:

$$\begin{tikzcd}
& \mathcal{M}^{\pi\mathcal{T}} & \\
\mathcal{Q}(\pi(\alpha)) \arrow[ur, "i^{\pi\mathcal{T}}"] & & \mathcal{M}^{\mathcal{T}} \arrow[ul, "\sigma"'] \\
& \mathcal{Q}^{\pi}(\alpha) \arrow[ul, "\pi"] \arrow[ur, "i^{\mathcal{T}}"'] &
\end{tikzcd}$$

Here σ is the copy map. Now let $Z \prec H_\eta^{\mathrm{HOD}_X}$ be of size β with $\beta \subset Y$ and everything relevant in it. Let $\tau : N \to Z$ be the reverse of the Mostowski collapse. Let $\mathcal{T}^* := \tau^{-1}(\pi\mathcal{T}) \in N[h]$ and $\mathcal{Q}^* := \tau^{-1}(\mathcal{Q}(\pi(\alpha)))$, identifying τ with its extension to $\mathrm{HOD}_X[h]$. On the one hand by elementarity we have $\mathcal{T}^* = (\tau^{-1} \circ \pi)\mathcal{T}$ and hence:

$$\begin{tikzcd}
& \mathcal{M}^{\mathcal{T}^*} & \\
\mathcal{Q}^* \arrow[ur, "i^{\mathcal{T}^*}"] & & \mathcal{M}^{\mathcal{T}} \arrow[ul, "\sigma^*"'] \\
& \mathcal{Q}^{\pi}(\alpha) \arrow[ul, "\tau^{-1}\circ\pi"] \arrow[ur, "i^{\mathcal{T}}"'] &
\end{tikzcd}$$

On the other hand $\mathcal{T}^*$ is by $\tau^{-1}(\Lambda_{\pi(\alpha)})$ which is just $\Lambda^{\tau}_{\pi(\alpha)} \restriction N[h]$. (Let $\mathcal{U}$ be by $\tau^{-1}(\Lambda_{\pi(\alpha)})$, then $\tau(\mathcal{U})$ is by Λ and $\tau\mathcal{U}$ is a hull of it. By hull condensation $\tau\mathcal{U}$ is by Λ.) Now τ is actually an iteration map, as $\mathcal{Q}^*$ is a tail of $(\mathcal{P}, \Sigma)$. Because of pullback consistency for HOD-pairs we'll have that $\mathcal{T}^*$ is actually by the tailstrategy of $(\mathcal{P}, \Sigma)$. Iteration maps commute so we have:

$$\begin{tikzcd}
\mathcal{Q}(\pi(\alpha)) & & \\
& & \mathcal{M}^{\mathcal{T}^*} \arrow[ull, "\sigma^{**}"'] \\
& \mathcal{Q}^* \arrow[uul, "\tau"] \arrow[ur, "i^{\mathcal{T}^*}"'] &
\end{tikzcd}$$

Here $\sigma^* * : \mathcal{M}^{\mathcal{T}^*} \to \mathcal{Q}(\pi(\alpha))$ is the iteration embedding. Putting things together we have:

$$\begin{tikzcd}
\mathcal{Q}(\pi(\alpha)) & & \mathcal{M}^{\mathcal{T}^*} \arrow[ll, "\sigma^{**}"'] & \\
& \mathcal{Q}^* \arrow[ul, "\tau"] \arrow[ur, "i^{\mathcal{T}^*}"] & & \\
& & & \mathcal{M}^{\mathcal{T}} \arrow[uul, "\sigma^*"'] \\
& & \mathcal{Q}^{\pi}(\alpha) \arrow[uuull, "\pi"] \arrow[uul, "\tau^{-1}\circ\pi"'] \arrow[ur, "i^{\mathcal{T}}"] &
\end{tikzcd}$$

Then $\sigma^{**} \circ \sigma^*$ is as wanted. $\dashv$

Lemma 6.3: *Let Y be a good hull, $\pi : M \to Y$ be the reverse of the Mostowski collapse. Let $\alpha < \bar{\lambda}$. Then $\mathcal{Q}^\pi||((\delta_\alpha^{\mathcal{Q}^\pi})^+)^{\mathcal{Q}^\pi} = \mathrm{I}^{\Lambda^\pi_{\mathcal{Q}(\pi(\alpha))}}(\mathcal{Q}^\pi||\delta_\alpha^{\mathcal{Q}^\pi})$.*

Proof: Inclusion follows easily from Lemma 6.1. To show the opposite direction we have to first realize that $\mathrm{I}^{\Lambda^\pi_{\mathcal{Q}(\pi(\alpha))}}(\mathcal{Q}^\pi||\delta_\alpha^{\mathcal{Q}^\pi}) \in M$. By Lemma 3.1 we have $\mathrm{Lp}^{\Lambda^\pi}(\bar{A}\cup\{\mathcal{Q}^\pi\}) \in M$. The rest then follows from the following claim:

Claim 1: $\forall\alpha < \bar{\lambda} : \mathrm{I}^{\Lambda^\pi_{\mathcal{Q}(\pi(\alpha))}}(\mathcal{Q}^\pi||\delta_\alpha^{\mathcal{Q}^\pi}) \subseteq \mathrm{I}^{\Lambda^\pi}(\mathcal{Q}^\pi)$.

Proof of Claim: Note that Λ^π has branch condensation by Lemma 2.46 and determines itself on generic extensions. We can thus do a core model induction relative to Λ^π. To that end take some Y that is ω-cofinal in $\mathrm{Lp}^{\Lambda^\pi_{\mathcal{Q}(\pi(\alpha))}}(A)$ where A codes $V_\kappa^{\mathrm{HOD}_X}$.

For some appropriate $\nu < \kappa$ we can then work in $\mathrm{HOD}_{X,Y}[g]$ where $g \subset \mathrm{Col}(\omega,\nu)$ is generic over V and form $S := L((\mathrm{Lp}^{+,\Lambda^\pi})^{\mathrm{HOD}_{X,Y}[g]}(\mathbb{R}^{\mathrm{HOD}_{X,Y}[g]}, \Lambda^\pi \restriction \mathbb{R}^{\mathrm{HOD}_{X,Y}[g]})$, also let $(\mathcal{P},\Sigma)$ be an ω-suitable pair relative to S just like the one we constructed in section 3.

We do have as in previous arguments that $\mathrm{Lp}^{S,\Lambda^\pi}(a) = \mathrm{I}^{\Lambda^\pi}(a)$ for all $a \in H_{\omega_1}^{\mathrm{HOD}_{X,Y}[g]}$. If we also had $\mathrm{Lp}^{S,\Lambda^\pi_{\mathcal{Q}(\pi(\alpha))}}(a) = \mathrm{I}^{\Lambda^\pi_{\mathcal{Q}(\pi(\alpha))}}(a)$ for all $\alpha < \bar{\lambda}$ we would be finished by mouse capturing.

Unfortunately, we do not see an abstract reason why this would hold, so we do have to do a little bit more work. Fix an $\alpha < \bar{\lambda}$, let $Y', \nu', g', (\mathcal{P}',\Sigma')$ be as above but for $\Lambda^\pi_{\mathcal{Q}(\pi(\alpha))}$. Let Z code the triple $\{X, Y, Y'\}$ and let $nu^* \geq \nu, \nu'$. Work in $\mathrm{HOD}_Z[h]$ where $h \subset \mathrm{Col}(\omega,\nu^*)$ is generiv over V.

Σ, Σ' extend to $\mathrm{HOD}_Z[h]$ so we can form the derived models of $\mathcal{P}$ and $\mathcal{P}'$ there, call them D and D' respectively. We must have $D \subseteq D'$ or $D' \subseteq D$. If the latter holds we have

$$\mathrm{I}^{\Lambda^\pi_{\mathcal{Q}(\pi(\alpha))}}(a) = \mathrm{Lp}^{D',\Lambda^\pi_{\mathcal{Q}(\pi(\alpha))}}(a) \subseteq \mathrm{Lp}^{D,\Lambda^\pi}(a) = \mathrm{I}^{\Lambda^\pi}(a)$$

where $a \in b \in H_{\omega_1}^{\mathrm{HOD}_Z[h]}$. As this includes $a = \mathcal{Q}^\pi||\delta_\alpha^{\mathcal{Q}^\pi}$ and $b = \mathcal{Q}^\pi$ we are done.

So, assume $D \subseteq D'$. We then have that $\Lambda^\pi \restriction \mathbb{R}^{\mathrm{HOD}_Z[h]} \in D'$ and hence

$$\mathrm{I}^{\Lambda^\pi_{\mathcal{Q}(\pi(\alpha))}}(a) = \mathrm{Lp}^{D',\Lambda^\pi_{\mathcal{Q}(\pi(\alpha))}}(a) \subseteq \mathrm{Lp}^{D',\Lambda^\pi}(a)$$

but any Λ^π-mouse in D' is in the derived model of the generically iterable $\mathcal{P}'$ and is therefore itself generically iterable as desired. □

It is now enough to show that $\mathrm{I}^{\Lambda^\pi_{\mathcal{Q}(\pi(\alpha))}}(\mathcal{Q}^\pi||\delta^{\mathcal{Q}^\pi}_\alpha)$ satisfies the definition of $\mathcal{Q}^\pi||((\delta^{\mathcal{Q}^\pi}_\alpha)^+)^{\mathcal{Q}^\pi}$ given by Lemma 6.1. So let $\tau : \bar{\mathcal{M}} \to \mathcal{M}$ be an appropriate hull where $\mathcal{M} \trianglelefteq \mathcal{Q}^\pi||((\delta^{\mathcal{Q}^\pi}_\alpha)^+)^{\mathcal{Q}^\pi}$. Clearly, we have that $\bar{\mathcal{M}} \trianglelefteq \mathrm{I}^{(\Lambda^\pi_{\mathcal{Q}(\pi(\alpha))^-})^\tau}(\tau^{-1}(\mathcal{Q}^\pi||\delta^{\mathcal{Q}^\pi}_\alpha))$. We just have to show that M recognizes the fact that $\bar{\mathcal{M}}$ is generically $(\mathrm{On}, \mathrm{On})$-iterable.

This works just like in the proof of Lemma 4.1. We just have to show that $(\Lambda^\pi)^\tau$ is OD in Λ^π from some element coded into $\bar{A}$. Of course τ itself is not coded into $\bar{A}$. But we can write $\tau = \sigma \circ \tau'$ where τ' is coded into $\bar{A}$ and σ is definable over it. σ here is just the preimage of some $\sigma^X_{\beta,<\kappa}$ for appropriate β. τ' comes from taking a hull and then realizing into $\mathcal{Q}^X_\alpha$. We skip further detail. ⊣

Lemma 6.4: *Let Y be a good hull, $\pi : M \to Y$ be the reverse of the Mostowski collapse. Let $\alpha < \bar{\lambda}$. Then, $\Lambda^\pi_{\mathcal{Q}(\pi(\alpha))}$ has branch condensation, determines itself on generic extensions and is fullness preserving for $<\kappa$-trees absolute to $<\kappa$-generic extensions.*

Proof: Branch condensation follows from Lemma 2.46, generic extensions is simply because it is the pullback of a generic iteration strategy. We will show fullness preservation now.

Assume now that $\mathcal{T}$ is a counterexample to fullness preservation, i.e.

- $\mathcal{T} \in \mathrm{HOD}_X[h]$ and is countable there where $h \subset \mathrm{Col}(\omega, \nu)$ is generic over V for some $\alpha < \kappa$;
- $\mathcal{T}$ is by Λ^π_α;
- $\mathcal{T}$ has a last model $\mathcal{Q}^*$, there is no drop on the main branch and for some $\beta \leq \lambda^{\mathcal{Q}^*}$ and some cutpoint γ of $\mathcal{Q}^*$, there exists some $\mathcal{M}$ s.t. $\mathcal{M} \trianglelefteq \mathrm{I}^{(\Lambda^\pi_\alpha)_{(\mathcal{Q}^*(\beta))^-}}(\mathcal{Q}^*||\gamma)$ but $\mathcal{M} \notin \mathcal{Q}^*$.

By the results of the previous section we can find some $(\mathcal{R}, \Phi)$ a Λ^π_α-HOD-pair s.t. Φ has branch condensation, determines itself on generic extensions and is fullness preserving. Because Φ is fullness preserving we can make $\mathcal{T}$ generic over some iterate of Φ and that iterate will correctly identify the missing mouse. Hence $\mathcal{R}$ believes: "In my derived model there exists a witness to the fact that Λ^π_α is not fullness preserving." We will also need $M^{\Phi,\#}_\omega$.

Note that no level of $\mathcal{R}$ projects across $\mathcal{Q}^\pi(\alpha)$ by Lemma 6.3. Therefore we can form the long ultrapower $\mathrm{Ult}(\mathcal{R}, \pi \restriction \mathcal{Q}^\pi(\alpha))$ which by countable closure of π is wellfounded. Similarly, we can form the long ultrapower $\mathrm{Ult}(\mathcal{R}, i^\mathcal{T})$, this too is well-founded because it realizes into the previous ultrapower by the previous lemma.

Let us now take $Z \prec H^{\mathrm{HOD}_X}_{\eta^*}$ be countable with everything relevant in it. Let $\tau : N \to Z$ be the reverse of the Mostowski collapse. Let $h \subset \mathrm{Col}(\omega, \tau^{-1}(\nu))$ be generic over N, and let $\mathcal{T}^* \in N[h]$ be a tree s.t.

$$N[h] \models \mathcal{T}^* \text{ witnesses a failure of } \tau^{-1}(\Lambda^\pi_\alpha) \text{ to be fullness preserving.}$$

Notice we have $\tau^{-1}(\Phi) = \Phi^\tau \upharpoonright N$, for conveniences sake we will confuse them from now on. Write $\tau^{-1}(\mathcal{R}) = \mathcal{R}^\tau$.

Let M^* be an iterate of $M_\omega^{\Phi^\tau,\#}$ at its bottom Woodin cardinal making $\mathcal{T}^*$ generic. We can assume that $M^* \in N$ by only making a name generic. We'll have that $M^*[\mathcal{T}^*]$ believes "in my derived model $\mathcal{T}^*$ witnesses a failure of $(\Lambda^\pi_\alpha)^\tau$ to be fullness preserving". We use here that the terms for the strategy of $\tau^{-1}(\mathcal{Q}^\pi(\alpha)$ will be interpreted as $(\Lambda^\pi_\alpha)^\tau$. We'll have that an iteration strategy for the missing mouse is Wadge reducible to Φ^τ.

Let now $\sigma : \mathcal{M}^{\mathcal{T}^*} \to \tau^{-1}(\mathcal{Q})$ s.t. $\tau^{-1}(\pi) = \sigma \circ i^{\mathcal{T}}$. Let $M^{**} := \mathrm{Ult}(M_\omega^{\Phi^\tau,\#}; i^{\mathcal{T}^*})$. It is embeddable into $\mathrm{Ult}(M_\omega^{\Phi^\tau,\#}; \tau^{-1}(\pi))$ and hence, by countable completeness, into $M_\omega^{\Phi,\#}$ by some σ^*.

Let D be the derived model of M^{**}. We then have $\Phi^\tau \in D$, as it is computable from Φ^{σ^*} which is the interpretation of M^{**}'s internal strategy in D. Hence D also contains the mouse missing from $\mathcal{M}^{\mathcal{T}^*}$. As this mouse is definable it must be in M^{**} and because of acceptability in $\mathcal{M}^{\mathcal{T}^*}$. Contradiction! $\dashv$

Lemma 6.5: *$\alpha + \omega < \lambda^X_\mu$ for all $\alpha < \lambda^X_\mu$.*

Proof: Notice that by the results of the preceding section this is certainly true for λ. We will now reflect this downwards.

Let α be as above. Let $(\mathcal{P}, \Sigma)$ be a HOD-pair s.t. Σ has branch condesation, determines itself on generic extensions and is fullness preserving for $<\kappa$-trees absolute to $<\kappa$-generic extensions, and $(\mathcal{P}, \Sigma)$ generates $\mathcal{Q}^X_\mu(\alpha)$. But, of course, $(\mathcal{P}, \Sigma)$ also generates $\mathcal{Q}(\sigma^X_\mu(\alpha))$.

Let X be a good hull, $\pi : M \to X$ the reverse of the Mostowski collapse. Because $\mathcal{P}$ has size μ we'll have that $(\mathcal{Q}^\pi(\pi^{-1}(\sigma^X_\mu(\alpha)), \Lambda^\pi_{\pi^{-1}(\sigma^X_\mu(\alpha)})$ is a tail of $(\mathcal{P}, \Sigma)$. By the above lemma we then have that $(\mathcal{Q}^\pi(\pi^{-1}(\sigma^X_\mu(\alpha) + \omega), \Lambda^\pi_{\pi^{-1}(\sigma^X_\mu(\alpha)+\omega})$ is as wanted. $\dashv$

Lemma 6.6: *Let Y be a good hull, $\pi : M \to Y$ be the reverse of the Mostowski collapse. Then $(\mathrm{I}^{(\Lambda^X_\mu)^\pi)})^M(a) = \mathrm{I}^{(\Lambda^X_\mu)^\pi)}(a)$ for all $a \in V^{M[g]}_{\bar{\kappa}}$.*

Proof: Note that the sequence $\langle \mathcal{Q}_\alpha : \alpha < \kappa\rangle$ is OD in $J_1(A)$, and so are the direct limit embeddings $\langle \sigma_{\alpha,\beta} : \alpha < \beta < \kappa\rangle$ between them and $\langle \sigma_{\alpha,<\kappa} : \alpha < \kappa\rangle$ into $\mathcal{Q}$. Therefore Λ_α too is OD in Λ and A, as by pullback consistency for HOD-pairs it is the pullback of Λ under the direct limit embedding.

Fix some $\mu < \alpha$ that is Θ-closed. We'll have that Λ^X_μ is computable from Λ_α and $\sigma^X_{\mu,\alpha}$. Note that $\sigma^X_{\mu,\alpha} \in V_\kappa^{\mathrm{HOD}_X[g]}$. W.l.o.g. $\sigma^X_{\mu,\alpha} \in Y$.

We now have that $(\Lambda^X_\mu)^\pi$ is computed in $M[g]$ from Λ^π and $\pi^{-1}(\sigma_{\alpha,<\mu} \circ \sigma^X_{\mu,\alpha})$. Crucially, the aforementioned embedding is actually an iteration embedding.

As we have that $\pi^{-1}(\sigma^X_{\mu,\alpha}) \in A[g]$ and $\pi^{-1}(\sigma_{\alpha,<\mu})$ definable over $\mathrm{Lp}^{\Lambda^\pi}(A)$ we get

$$\mathrm{Lp}^{(\Lambda^X_\mu)^\pi}(A[g]) \subset \mathrm{Lp}^{\Lambda^\pi}(A[g]) = \mathrm{Lp}^{\Lambda^\pi}(A)[g] \in M[g].$$

The rest is as in Lemma 4.1. $\dashv$

Let now $\mathcal{P} := \mathrm{I}^{\Lambda^X_\mu}(\mathcal{Q}^X_\mu)$. Let Y be some good hull, $\pi : M \to Y$ be the reverse of the Mostowski collapse. Write $\pi^{-1}(\mathcal{P}) := \mathcal{P}_Y$, $\Sigma_Y := (\Lambda^X_\mu)^\pi$ and δ_Y for the supremum of Woodin cardinals in $\mathcal{P}_Y$. We will define an iteration strategy Σ^+_Y for $\mathcal{P}_Y$ s.t. for any $<\kappa$-iteration tree $\mathcal{T}$ with a last model and no drop on the main branch we have $\sigma : \mathcal{M}^\mathcal{T} \to \mathcal{Q}$ s.t. $\sigma^X_{\mu,<\kappa} \circ \pi = \sigma \circ i^\mathcal{T}$: let $\mathcal{U}$ be a normal component on some $M \in \mathcal{T}$ s.t. $\sigma_M : M \to \mathcal{Q}$ as above exists and $\mathcal{U}$ is based on some $M(\alpha+1)$, then $\mathcal{U}$ is by $\Lambda^{\sigma_M(\alpha+1)}_{\sigma_M(\alpha+1)}$, the realization embedding on $\mathcal{M}^\mathcal{U}$ comes from taking a hull of the copied tree on $\mathcal{Q}$; in the case of drops we pick the branch with the appropriate Q-structures; we skip further details.

We want to show that no initial segment projects across $\mathcal{Q}^X_\mu$. Assume not: Let Y be some good hull, $\pi : M \to Y$ be the reverse of the Mostowski collapse. Let $\beta < \lambda^{\mathcal{P}_Y}$ be s.t. $\rho_\omega(\mathcal{P}_Y) < \delta^{\mathcal{P}_Y}_\beta$ and $\mathrm{cof}^{\mathcal{P}_Y}(\delta^Y) < \delta^{\mathcal{P}_Y}_\beta$. Σ^+_Y does determine itself on generic extensions so it is contained in some determinacy model D, though we might have to move to a larger universe to do so. Let $\bar{Q}$ be the core of $\mathcal{P}_Y$ above $\delta^{\mathcal{P}_Y}_\beta$ and let a be the new set defined over it. We then have $a \in \mathrm{HOD}^D_{(\Sigma_Y)_\beta}$ on the other hand $(\Sigma_Y)_{\beta+1}$ is fullness preserving, so $\mathcal{P}_Y(\beta+1)$ will iterate into a cardinal inital segment of $\mathrm{HOD}^D_{(\Sigma_Y)_\beta}$ meaning it actually does contain a. Contradiction!

Lemma 6.7: *Let Y be a good hull, $\pi : M \to Y$ be the reverse of the Mostowski collapse. Σ^+_Y is fullness preserving.*

PROOF: We can assume that $\eta := \mathrm{cof}^{\mathcal{P}_Y}(\delta^Y)$ is measurable in there, otherwise we can proceed just as in Lemma 6.4. We want to show that adding $\mathrm{I}^{\Sigma^+_Y}$ to $\mathcal{P}_Y$ will not project across. Otherwise let $\mathcal{M} \trianglelefteq \mathrm{I}^{\Sigma^+_Y}(\mathcal{P}_Y)$ s.t. $\rho(\mathcal{M}) \leq \delta^Y$. We can assume that $\mathcal{M}$ projects exactly to δ^Y, otherwise we can argue as above.

Let $f : \eta \to \delta^Y$ be cofinal, continuous and increasing. Let $\langle M_\xi : \xi < \eta \rangle$ be a sequence s.t. $M_\xi \subset \mathcal{P}_Y || f(\xi)$ codes the theory of $\mathcal{M}$ on ordinals $<f(\xi)$ and the standard parameter. Let U be the order 0 measure on $\mathrm{cof}^{\mathcal{P}_Y}(\delta^Y)$. Let $\mathcal{P}^* := \mathrm{Ult}(\mathcal{P}_Y; U)$ (this is the appropriate fine structural ultrapower) and j the ultrapower embedding. Note that j acts on $\langle M_\xi : \xi < \eta \rangle$.

We have that $\Sigma^* := \bigoplus_{\alpha < \delta_Y} \Sigma^{\mathcal{P}^*}_{j(\alpha)}$ is OD in Σ_Y, because it is computable from U. Hence, by mouse capturing,

$$\mathcal{P}^{\mathcal{P}^*}(\delta^Y) \subseteq \mathrm{I}^{\Sigma^*}(\mathcal{P}^* || \delta^Y) \subset \mathrm{I}^{\Sigma_Y}(\mathcal{P} || \delta^Y).$$

Now $M^* := j(\langle M_\xi : \xi < \eta \rangle)(\delta^Y)$ is coded as a subset of δ^Y and is therefore in $\mathcal{P}$. But we can compute the theory of $\mathcal{M}$ from it and $j \restriction \delta^Y$. Contradiction!

From here on out we can proceed just like in Lemma 6.4. $\dashv$

Definition 6.8: Let Y be a good hull, $\pi : M \to Y$ be the reverse of the Mostowski collapse. Let $A \in \mathcal{P}^{\mathcal{P}_Y}(\delta_Y)$ and φ be a first order formula with two free variables. We say Y has (φ, A) condensation if for all countable (in $\mathrm{HOD}_X[g]$) $\mathcal{R}$ together with elementary embeddings $\nu : \mathcal{P}_Y \to \mathcal{R}$ and $\tau : \mathcal{R} \to \mathcal{P}$ s.t. $\nu \restriction \delta_Y$ is an iteration embedding according to Σ_Y, then $\nu(T^\varphi_{\mathcal{P}_Y,A}) = T^\varphi_{\mathcal{R},\tau,A}$ where

$$T^\varphi_{\mathcal{P}_Y,A} = \{s \in [\delta_Y]^{<\omega} | \mathcal{P}_Y \models \varphi(s, A)\}$$

and

$$T^{\varphi}_{\mathcal{R},\tau,\pi(A),\mathcal{P}} = \{s \in [\nu(\delta_Y)]^{<\omega} | \mathcal{P} \models \varphi(\pi^{\Sigma^{\tau,-}_{\mathcal{R}}}_{\mathcal{R}(\alpha_s),\infty}(s), \pi(A))\}$$

where $\alpha_s < \lambda^{\mathcal{R}}$ is minimal with $s \in \mathcal{R}(\alpha_s)$ and $\pi^{\Sigma^{\tau,-}_{\mathcal{R}}}_{\mathcal{R}(\alpha_s),\infty}$ is the iteration embedding from $\mathcal{R}(\alpha_s)$ into $\mathcal{P}$ by the τ-pullback strategy.

Lemma 6.9 (Sargsyan-Trang): *There is a good hull Y that has (φ, A)-condensation for all φ and $A \in \mathcal{P}^{\mathcal{P}_Y}(\delta_Y)$.*

Proof: Just as in [Tra], but for the reader's convenience we will reproduce the argument here: let us assume for a contradiction that there exists a stationary set S of good hulls Y s.t. there exist a first order formula φ_Y and $A_Y \subset \mathcal{P}^{\mathcal{P}_Y}(\delta_Y)$ that are a counterexample to (φ_Y, A_Y)-condensation.

We say $\langle \varphi, Y_i, Z, \mathcal{R}_i, A_i, \nu_i, \tau_i, \sigma_i : i < \omega\rangle$ is a bad tuple iff

- Y_i is a good hull for all $i < \omega$, if $i < j$ then $Y_i \subset Y_j$, write $\pi_{i,j} : \mathrm{tc}(Y_i) \to \mathrm{tc}(Y_j)$ for the canonical embedding;
- Z is a good hull s.t. $\langle Y_i, \mathcal{R}_i, A_i, \nu_i, \tau_i : i < \omega\rangle \in Z$;
- $\nu_i : \mathcal{P}_{Y_i} \to \mathcal{R}_i$ and $\tau_i : \mathcal{R}_i \to \mathcal{P}_{Y_{i+1}}$ are elementary, $\pi_{i,i+1} = \tau_i \circ \nu_i$, $\nu_i \restriction \delta_Y$ is the iteration embedding according to Σ_{Y_i} for all $i < \omega$;
- $A_i \subset \mathcal{P}^{\mathcal{P}_{Y_i}}(\delta_{Y_i})$ for all $i < \omega$, if $i \leq j$ then $\pi_{i,j}(A_i) = A_j$;
- $\sigma_i : \mathcal{P}_{Y_i} \to \mathcal{M}^Z_\infty$ where $\mathcal{M}^Z_\infty$ is the direct limit of all Σ_Z-iterates for all $i < \omega$, if $i \leq j$ then $\tau_i = \tau_j \circ \pi_{i,j}$;
- $\nu_i(T^{\varphi}_{\mathcal{P}_{Y_i},A_i}) \neq T^{\varphi}_{\mathcal{R}_i,\tau_i,\sigma_i(A_i),\mathcal{M}^Z_\infty}$ for all $i < \omega$.

Claim 1: *There exists a bad tuple.*

Proof of Claim: By pressing down we can find a stationary set S^* of good hulls Y s.t. $(\varphi_Y, \pi_Y(A_Y))$ is constant where $\pi_Y : M_Y \to Y$ is the reverse of the Mostowski collapse. Let (φ, A) be this constant value. Let Y_i be an ascending sequence in S^* s.t. a witness $(\mathcal{R}_i, \nu_i, \tau_i)$ for the failure of $(\varphi, \pi_{Y_i}^{-1}(A))$-condensation is in Y_{i+1}. Set $A_i := \pi_{Y_i}^{-1}(A)$. Let Z be a good hull s.t. $\langle Y_i, \mathcal{R}_i, \nu_i, \tau_i\rangle \in Z$, let $\pi : N \to Z$ be the reverse of the Mostowski collapse. By elementarity and the failure of condensation we have

$$\nu_i(T^{\varphi}_{\mathcal{P}_{Y_i}, A_i}) \neq \{s \in [\nu_i(\delta_{Y_i})]^{<\omega} | \mathcal{P}_Z \models \varphi(\pi^{\Sigma^{\tau,-}_{\mathcal{R}}}_{\mathcal{R}(\alpha_s), \mathcal{P}_Z}(s), \pi^{-1}(A))\}$$

for all $i < \omega$ where $\pi^{\Sigma^{\tau,-}_{\mathcal{R}}}_{\mathcal{R}(\alpha_s), \mathcal{P}_Z}$ is the embedding given by the direct limit of all $(\mathcal{P}, \Sigma)$ in $Z\,[g]$. Let then $\iota : \mathcal{P}_Z \to \mathcal{M}^Z_\infty$ be the direct limit embedding. It is then easy to see that $\langle \varphi, Y_i, Z, \mathcal{R}_i, A_i, \nu_i, \tau_i, \iota \circ \pi^{-1} \circ \pi_{Y_i} : i < \omega\rangle$ is as wanted. $\square$

So let $\mathcal{A} := \langle \varphi, Y_i, Z, \mathcal{R}_i, A_i, \nu_i, \tau_i, \sigma_i : i < \omega\rangle$ is a bad tuple. We can find $(\mathcal{R}, \Phi) \in \mathrm{HP}^{\mathrm{HOD}_X}_{\Sigma_{Y_i}}(\mu)$ s.t. Φ has branch condensation, determines itself on generic extensions, is fullness preserving for $<\kappa$-trees absolute to $<\kappa$-generic extensions, and

$$L(\Phi, \mathbb{R}) \models \mathcal{A} \text{ is a bad tuple.}$$

Two things to note here: firstly, $\mathcal{A}$ is not hereditarily countable but it is coded by a real $x \in \mathrm{HOD}_X\,[g]$: x codes both $\langle \varphi, \mathcal{P}_{Y_i}, \mathcal{P}_Z, \mathcal{R}_i, \nu_i, \tau_i, (\sigma_i)' : i < \omega\rangle$ and a contionuous map f s.t. f^{-1}" $[\mathrm{Code}(\Phi)]$ codes a pair $(\mathcal{P}, \Sigma)$ which generates $\mathcal{M}^Z_\infty$, and if $\iota : P \to \mathcal{M}^Z_\infty$ is the direct limit embedding then $\sigma_i = \iota \circ (\sigma_i)'$.

Secondly, in general we can not assume that $\mathcal{R}$ has size μ. But if not, then we can just replace $\mathcal{M}^Z_\infty$ by a larger direct limit $(\mathcal{M}^Z_\infty)^*$ appropriate to the size of $\mathcal{R}$. So, w.l.o.g. we can and do assume that $(\mathcal{R}, \Phi) \in \mathrm{HP}^{\mathrm{HOD}_X}(\mu)$.

Let now $\mathcal{W} := M^{\Phi,\#}_\omega$. Let $\mathcal{W}^*$ be the iterate that results from making a $\mathrm{Col}(\omega, \mu)$-name ρ for a real coding $\mathcal{A}$ generic at $\mathcal{W}$'s bottom Woodin cardinal. Then $\mathcal{W}^*\,[\rho]\,[g]$ believes "in my derived model ρ^g codes a bad tuple". Let $p \in \mathrm{Col}(\omega, \mu)$ be a condition that forces this.

Let now U be a countable hull that contains everything relevant, let $\mathfrak{o} : O \to U$ be the reverse of the Mostowski collapse. Let $\bar{\rho} = \mathfrak{o}^{-1}(\rho)$, etc. Let $\bar{p} \in \bar{g} \subset \mathrm{Col}(\omega, \bar{\mu})$ be generic over O. Let $\langle \varphi, \bar{Y}_i, \bar{Z}, \bar{\mathcal{R}}_i, \bar{A}_i, \bar{\nu}_i, \bar{\tau}_i, \bar{\sigma}_i : i < \omega\rangle$ be the preimages.

We now write $\bar{\mathcal{W}}_0 := \mathfrak{o}^{-1}(\mathcal{W}^*)$. Then we define inductively $\bar{\mathcal{U}}_i := \mathrm{Ult}(\bar{\mathcal{W}}_i; \bar{\nu}_i)$ writing ν_i^* for the ultrapower embedding and $\bar{\mathcal{W}}_{i+1} := \mathrm{Ult}(\mathcal{U}_i; \bar{\tau}_i)$ writing τ_i^* for the ultrapower embedding.

We see that each of these ultrapowers realizes into $\mathrm{Ult}(\bar{\mathcal{W}}_0; \pi_{\bar{Y}_0})$ which in turn by countable completeness of Y_0 realizes into $\mathcal{W}^*$. Let $\alpha_i : \bar{\mathcal{W}}_i \to \mathcal{W}^*$ and $\beta_i : \bar{\mathcal{U}}_i \to \mathcal{W}^*$ be the realization embeddings. We then have that $\bar{\mathcal{W}}_i$ is a $\bar{\Phi}_i := \Phi^{\alpha_i}$-mouse and $\bar{\mathcal{U}}_i$ is a $\bar{\Psi}_i := \Phi^{\beta_i}$-mouse for all $i < \omega$. It is important to note that $\bar{\Phi}_i = \bar{\Psi}_i^{\nu_i^*}$ and $\bar{\Psi}_i = \bar{\Phi}_{i+1}^{\tau_i^*}$ for all $i < \omega$.

Now let $j_n : \bar{\mathcal{W}}_n \to \bar{\mathcal{W}}_n^*$ and $k_n : \bar{\mathcal{U}}_n \to \bar{\mathcal{U}}_n^*$ be the result of a simultaneous $\mathbb{R} \cap \mathrm{HOD}_X\,[g]$ - hereafter just $\mathbb{R}$ - genericty iteration (see proof of Lemma 4.8). Let us write $l_n : \bar{\mathcal{W}}_n^* \to \bar{\mathcal{U}}_n^*$ and $m_n : \bar{\mathcal{U}}_n^* \to \bar{\mathcal{W}}_{n+1}^*$ for the copy maps.

Let $C_n = L(\bar{\Phi}_n, \mathbb{R})$ and $D_n = L(\bar{\Psi}_n, \mathbb{R})$ be the derived models of $\bar{\mathcal{W}}_n^*$ and $\bar{\mathcal{U}}_n^*$ respectively. We have that $C_n \subseteq D_n$ and $D_n \subseteq C_{n+1}$ for all $n < \omega$.

Our witness to "badness" is not $(\mathcal{M}_\infty^Z, \sigma_0(A_0))$ but instead if $((\bar{\mathcal{P}}, \bar{\Sigma}), A')$ is a preimage of it in $\bar{\mathcal{W}}_0 [\bar{\rho}] [\bar{g}]$, then the HOD-limit $(\mathcal{M}_\infty^Z)^*$ of $(\bar{\mathcal{P}}, \bar{\Sigma})$ as computed in C_0 and A^* the image of A' under the direct limit embedding is our witness, i.e. $\bar{\nu}_i(T^\varphi_{\mathcal{P}_{\bar{Y}_i}, \bar{A}_i}) \neq T^\varphi_{\bar{\mathcal{R}}_i, \bar{\tau}_i, A^*, (\mathcal{M}_\infty^Z)^*}$.

The pair $((\mathcal{M}_\infty^Z)^*, A^*)$ is definable in each of the C_n and D_n. $\mathcal{M}_\infty^Z$ is just the HOD of some common Wadge initial segment of each of the C_n, D_n and A^* can be defined from its position in the canonical well-order of that HOD. Let t be a parameter defining that pair, and let $\theta(\cdot, \cdot)$ be a first oder formula s.t.

$$\mathcal{X} \models \theta(s, t) \Leftrightarrow (\mathcal{M}_\infty^Z)^* \models \varphi(s, A^*)$$

where $\mathcal{X}$ can be any of the models C_n, D_n.

Taking stock, we have that

$(1)_n$ $s \in T^\varphi_{\mathcal{P}_{\bar{Y}_i}, \bar{A}_i}$ iff $C_n \models \theta(\pi^{\Sigma_{\bar{Y}_n}}_{\mathcal{P}_{\bar{Y}_n}, \infty}(s), t)$ for all $s \in [\delta_{\bar{Y}_n}]^{<\omega}$.

Here $\pi^{\Sigma_{\bar{Y}_n}}_{\mathcal{P}_{\bar{Y}_n}, \infty} : \mathcal{P}_{\bar{Y}_n} \to (\mathcal{M}_\infty^Z)^*$ is the map given by the HOD-limit of $(\mathcal{P}_{\bar{Y}_n}, \Sigma_{\bar{Y}_n})$ where $\Sigma_{\bar{Y}_n}$ is simply the (secondary) strategy of $\mathcal{P}_{\bar{Y}_n}$ on the sequence of $\bar{\mathcal{W}}_n^*$.

Thus, it should be easy to see that this constitutes a first order statement over $\bar{\mathcal{W}}_n^*$. On the other hand using the "badness" of our tuple we get:

$(2)_n$ there exists $s \in \bar{\nu}_n(T^\varphi_{\mathcal{P}_{\bar{Y}_i}, \bar{A}_i})$ s.t. $D_n \models \neg\theta(\pi^{\Sigma^{\bar{\tau}_n, -}_{\bar{\mathcal{R}}_n}}_{\bar{\mathcal{R}}_n(\alpha_s), \infty}(s), t)$.

Notice here that $\Sigma^{\bar{\tau}_n, -}_{\bar{\mathcal{R}}_n}$ is the (secondary) iteration strategy on the sequence of $\bar{\mathcal{U}}_n^*$. Now, the direct limit of

$$\bar{\mathcal{W}}_0^* \to_{l_0} \bar{\mathcal{U}}_0^* \to_{m_0} \bar{\mathcal{W}}_1^* \dots$$

is well-founded as it can be embedded into an iterate of $\mathcal{W}^*$. Therefore we can find some $n^* < \omega$ s.t. l_n, m_n fix t for all $n > n^*$. Let then n be such. By elementarity of l_n we have

$(3)_n$ $s \in \bar{\nu}_i(T^\varphi_{\mathcal{P}_{\bar{Y}_i}, \bar{A}_i})$ iff $D_n \models \theta(\pi^{\Sigma^{\bar{\tau}_n, -}_{\bar{\mathcal{R}}_n}}_{\bar{\mathcal{R}}_n(\alpha_s), \infty}(s), t)$ for all $s \in [\bar{\nu}_n(\delta_{Y_n})]^{<\omega}$.

But this clearly contradicts $(2)_n$! $\dashv$

Let now Y be as above, $\pi : M \to Y$ the reverse of the Mostowski collapse. Let $\nu : \mathcal{P}_Y \to \mathcal{R}$ be a countable (in $\mathrm{HOD}_X [g]$) non-dropping Σ_Y^+-iterate. We then have a realization embedding $\tau : \mathcal{R} \to \mathcal{P}$ as required by condensation. (While we usually realize into $\mathcal{Q}$, for countable trees the realization embedding will factorize through $\mathcal{P}$, as can be easily seen through a good-hull-reflection argument.)

Now define $\sigma : \mathcal{R} \to \mathcal{P}$ by sending $\nu(f)(a)$ to $\pi(f)(\pi^{(\Sigma_Y^+)_\mathcal{R}}_{\mathcal{R}(\alpha), \infty}(a))$ where $\alpha < \lambda^\mathcal{R}$ and $f \in \mathcal{P}_Y$. By condensation we have that σ is elementary and it should be easy to see that σ is the Σ_Y^+-iteration embedding below $\nu(\delta_Y)$.

Let now $\mathcal{M}_\infty$ be the direct limit of all countable non-dropping Σ_Y^+-iterates and $\nu_\infty : \mathcal{P}_Y \to \mathcal{M}_\infty$ the embedding. Let σ^* be the direct limit of the above embeddings. Then $\mathrm{crit}(\sigma^*) = \nu_\infty(\delta_Y)$, hence $\mathcal{M}_\infty \models$ "$\nu_\infty(\delta_Y)$ is regular". By elementarity $\mathcal{P}_Y \models$ "δ_Y is regular".

Now let M be some countable in HOD_X hull of $M_\omega^{\Sigma_Y^+,\#}$. M's iteration strategy determines itself on generic extensions, so we can form D the derived model of M in V. There exists then in D some $\Gamma \subset \mathcal{P}(\mathbb{R})$ s.t.

$$L(\Gamma, \mathbb{R}) \models \text{" } \mathrm{AD}_\mathbb{R} + \Theta \text{ is regular".}$$

7 Conclusion

Our proof can be easily adapted to some different situations:

Theorem 7.1: *Assume $V \models \mathrm{ZF}$ and all successor cardinals are weakly compact and all limit cardinals are singular. Then there exists an inner model containing all the reals that satifies $\mathrm{ZF} + \mathrm{AD}_\mathbb{R} +$"$\Theta$ is regular".*

PROOF: The crucial clue is that under these circumstances all putative square sequences are threadable. We thus have that $\mathrm{Lp}(A)$ has countable cofinality and so does the stack. The only significant change is in proofing an analagouge of Lemma 3.4: we will come across iteration trees whose cofinality is not countable. But then its cofinality must be weakly compact. Hence any such tree has a unique cofinal branch. $\dashv$

Theorem 7.2: *Let κ be a singular strong limit, and assume that $\square_\kappa$ fails. Then there exists an inner model containing all the reals that satifies $\mathrm{ZF} + \mathrm{AD}_\mathbb{R} +$"$\Theta$ is regular".*

PROOF: It is shown in [Sar14] how to get the "next θ". We can thus form $\mathcal{Q}_{<\kappa}$ as in the main body of the paper, we can completely ignore the superscripts here. One major difference is that our mice will only be κ^+-iterable in V instead of fully generically iterable.

Consider now $\mathrm{Lp}^{\Lambda_{<\kappa}}(\mathcal{Q}_{<\kappa})$ (skip subscripts from now on). Working in a model of choice we must re-interpret Lp to mean that countable hulls have (ω_1, ω_1+1)-strategies. We will have that $\mathrm{On} \cap \mathrm{Lp}^\Lambda(\mathcal{Q}) < \kappa^+$, hence $\mathrm{cof}(\mathrm{On} \cap \mathrm{Lp}^\Lambda(\mathcal{Q})) < \kappa$.

We can find some countably closed $\mu < \kappa$ s.t. $\mathrm{cof}(\mathrm{On} \cap \mathrm{Lp}^\Lambda(\mathcal{Q}))$. We can then define a notion of good hull as countably closed hulls that are cofinal in $\mathrm{On} \cap \mathrm{Lp}^\Lambda(\mathcal{Q})$. If $\pi : M \to Y$ then reverses the Mostowski collapse of a good hull we can look at $(\mathcal{Q}^\pi, \Lambda^\pi)$ and again we will have that $\Lambda^\pi_{\mathcal{Q}(\pi(\alpha))}$ is π-realizable and that $\mathrm{I}^{\Lambda^\pi_{\mathcal{Q}(\pi(\alpha))}}(\mathcal{Q}^\pi(\alpha)||\delta_\alpha^{\mathcal{Q}^\pi}) \subset \mathcal{Q}^\pi_\alpha$. Again $\mathrm{I}^{\Lambda^\pi_{\mathcal{Q}(\pi(\alpha))}}(\cdot)$ is to be interpeted as the stack of (κ^+, κ^+)-iterable hybrid mice.

We then want to conclude that $\Lambda^\pi_{\mathcal{Q}(\pi(\alpha))}$ is fullness preserving. Otherwise we could find some $\Lambda^\pi_{\mathcal{Q}(\pi(\alpha))}$-suitable pair $(\mathcal{R}, \varphi)$ being witness to a counterexample. Now, here we might a priori have that $\lambda^\mathcal{Q} = \omega$, but even if α is a successor $\mathcal{R}$ would add subsets to $\mathcal{Q}^\pi(\alpha)$ but we still have $\mathrm{I}^{\Lambda^\pi_{\mathcal{Q}(\pi(\alpha+1))}}(\mathcal{Q}^\pi(\alpha)) \subset M$. Therefore we can still lift π onto $\mathcal{R}$ and proceed as before.

Now we can proceed by looking at $\mathcal{P} := \mathrm{I}^{\Lambda_\mu}(\mathcal{Q}_\mu)$ as above, and see that $\lambda^\mathcal{P}$ is regular in $\mathcal{P}$. $\dashv$

We will leave the reader with a few questions:
Question 1: *Assume* ZF *and that all uncountable cardinals are singular. Does there exist an inner model containing all the reals, satisfying* $\mathrm{AD}^+ + \mathrm{LSA}$*?*

A possible approach might look like this: we form a K^c-like construction $\langle \mathcal{N}_\alpha : \alpha \leq \gamma \rangle$ on top of $\mathcal{P}$ s.t.

- $\mathcal{P} \trianglelefteq \mathcal{N}_\alpha$ and $\rho_\omega(\mathcal{N}_\alpha) > \mathrm{On} \cap \mathcal{P}$ for all α;
- for all good hulls Y with $\pi : M \to Y$ being the reverse of the Mostowski collapse, we have that $\mathcal{N}_\alpha^Y = \pi^{-1}(\mathcal{N}_\alpha)$ has a fullness preserving π-realization strategy Ψ_α^Y with branch condensation (sometime Ψ_α^Y will only be a short tree strategy);
- for all good hulls Y with $\pi : M \to Y$ being the reverse of the Mostowski collapse, for all E on the sequence of $\mathcal{N}_\alpha^Y$ s.t. $\mathrm{crit}(E) = \delta_Y$ we have that
$$(a, A) \in E \Leftrightarrow \pi^{\Psi_\beta^Y}_{\mathcal{N}_\beta^Y, \infty}(a) \in \pi(A)$$
where $\pi^{\Psi_\beta^Y}_{\mathcal{N}_\beta^Y, \infty} : \mathcal{N}_\beta^Y \to \mathcal{P}$ is the iteration embedding by Ψ_β^Y, β being the stage of the construction where $\pi(E)$ was added;
- let E be on the sequence of $\mathcal{N}_\alpha$ with $\mathrm{crit}(E)$ larger than the supremum of Woodin cardinals of $\mathcal{P}$, then E is certified by a collapse in $\mathrm{HOD}_X[g]$.

It has been shown that such a construction can succeed. In our case we see two problems:
Firstly, assume we have already constructed $\mathcal{N}_\alpha$ with the above properties. We do not know that given a good hull Y with $\pi : M \to Y$ being the reverse of the Mostowski collapse M is closed under $\mathrm{I}^{\Psi_\alpha^Y}$. Usually, the next step in the construction would be to let $\mathcal{N}_{\alpha+1}^*$ the stack of all Lp-type premice $\mathcal{M}$ over $\mathcal{N}_\alpha$ s.t. $\tau^{-1}(M) \trianglelefteq \mathrm{I}^{\Psi_\alpha^Z}(\mathcal{N}_\alpha^Z)$ for all but non-stationarily many good hulls $\tau : N \to Z$.
In our situation we do not know how to guarantee that $\pi^{-1}(\mathcal{N}_{\alpha+1}^*) = \mathrm{I}^{\Psi_\alpha^Y}(\mathcal{N}_\alpha^Y)$ for any good hull $\pi : M \to Y$. This is a problem as the next extender which is derived from Ψ_α^Y and π as above does only fit on a full premouse.
The second problem is simply that in our situation a K^c-like model is not enough. We need the generic absoluteness that only a proper core model can give us. But we do not know that $\mathcal{N}_\alpha$ as above has an $(\mathrm{On}, \mathrm{On})$-iteration strategy.
Conveniently, both of these problems have a commom solution: if we could show that $\Psi_\alpha^Y \restriction V^M_{\pi^{-1}(\kappa)} \in M$ for many good $\pi : M \to Y$, then we would by reflection have that $\mathcal{N}_\alpha$ is (κ, κ)-iterable, but also that it iterates into $\mathcal{Q}$. Then Ψ_α^Y becomes definable from Λ^π as a pullback and hence $\mathrm{I}^{\Psi_\alpha^Y} \restriction V^M_{\pi^{-1}(\kappa)} \in M$ just as in Lemma 6.6.
Unfortunately, we do not yet know how to prove this, but we think that a solution is not too far off. More difficult (more interesting?) questions would be:
Question 2: *Assume* ZF *and that all uncountable cardinals below* $\Theta(\omega)$ *are singular. Does there exist - possibly in a generic extension - an inner model of* AD*?*

QUESTION 3: *What is the consistency strength of "*ZF *and there exists a pair of successive cardinals* κ, κ^+ *which are both singular"?*

As we mentioned at the introduction, our methods are not suitable to the above questions. We will need a much finer approach.

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